Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 57 (2007) 777–797
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

A geometrical setting for geometric phases on complex Grassmann manifolds

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Received 27 July 2005; received in revised form 18 May 2006; accepted 9 June 2006
Available online 24 July 2006

Abstract

The geometry of Grassmann manifolds $\mathrm{Gr}_K(\mathfrak{H})$, of orthogonal projection manifolds $\mathcal{P}_K(\mathfrak{H})$ and of Stiefel bundles $\mathrm{St}(K, \mathfrak{H})$ is reviewed for infinite dimensional Hilbert spaces K and $\mathfrak{H}$. Given a loop of projections, we study Hamiltonians whose evolution generates a geometric phase, i.e. the holonomy of the loop. The simple case of geodesic loops is considered and the consistence of the geodesic holonomy group is discussed. This group agrees with the entire $U(K)$ if $\mathfrak{H}$ is finite dimensional or if $\dim(K) \leq \dim(K^{\perp})$. In the remaining case we show that the holonomy group is contained in the unitary Fredholm group $U_{\infty}(K)$ and that the geodesic holonomy group is dense in $U_{\infty}(K)$.

MSC: 58BXX; 58B25; 53C22; 53C29

Subj. Class.: Geometric methods in physics

Keywords: Grassmann manifolds; Geometric phases; Geodesics; Stiefel bundles; Holonomy

1. Introduction

Grassmann manifolds are a classical subject in differential and algebraic topology. The appearance of this topic in ordinary quantum physics is connected to the study of geometric phases, which are just holonomies of the canonical connection $\mathcal{A}$ on the Stiefel bundle [30,35]. For a review of the argument, see [29] and [5].

The evolution of a quantum system is described in a complex separable Hilbert space $\mathfrak{H}$ and it is governed by a curve $U(t)$ of unitary operators generated by a (possibly time-dependent) Hamiltonian. The curve $U(t)$ induces an evolution on the states of the system represented by the projective space of $\mathfrak{H}$, on the subspaces of $\mathfrak{H}$ and on their orthonormal frames, and hence on the Grassmann manifolds $\mathrm{Gr}_K(\mathfrak{H})$ and on the corresponding Stiefel bundles $\mathrm{St}(K, \mathfrak{H})$. In this paper $\mathrm{Gr}_K(\mathfrak{H})$ denotes the manifold of subspaces in $\mathfrak{H}$ which are isomorphic to a given Hilbert space K and $\mathrm{St}(K, \mathfrak{H})$ denotes the manifold of isometric embeddings of K in $\mathfrak{H}$. Of course, isometric embeddings are identified with orthonormal frames for subspaces in $\mathrm{Gr}_K(\mathfrak{H})$, if an orthonormal basis is fixed in K.

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0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2006.06.002

Suppose that a subspace $E \in \mathrm{Gr}_K(\mathfrak{H})$ undergoes a cyclic evolution for $t \in [0, T]$, i.e. $U(T)E = E$. At time T, every isometric embedding u_0 of K in $\mathfrak{H}$ with image E becomes $u(T) = U(T)u_0$, where $u(T)$ is again an isometric embedding of K in $\mathfrak{H}$ with image E. There exists a unique operator $\Phi \in U(K)$ such that $u(T) = u_0\Phi$; the operator Φ is called a geometric phase if it agrees with the holonomy of the loop at u_0. If Φ is a geometric phase, the evolution in E at the final time T does not depend on the specific Hamiltonian governing the evolution, but only on the loop (up to reparametrization) and on the geometry of $\mathrm{St}(K, \mathfrak{H})$.

The geometric phase is often achieved in adiabatic processes, as in Berry's pioneering work [3]. In this case its geometrical description is more involved. For example, in [37] a family of Hamiltonians is considered which depend on some control parameters. The Hamiltonians are supposed to have null eigenspaces with the same degeneracy, so that a map from the parameter manifold on a Grassmann manifold $\mathrm{Gr}_K(\mathfrak{H})$ is given. A loop in the parameter space gives a time-dependent Hamiltonian $H(t)$ and also a loop in $\mathrm{Gr}_K(\mathfrak{H})$. Under adiabatic approximation, the loop in $\mathrm{Gr}_K(\mathfrak{H})$ can be considered the evolution of the initial eigenspace in the dynamic governed by $H(t)$ and the corresponding operator Φ approximates the geometric phase. In this setting the geometry of the parameter manifold is relevant, since geometric phases are holonomies relative to the pullback of $\mathrm{Gr}_K(\mathfrak{H})$ on the parameter space.

A renewed interest for geometric phases in physics is motivated by the possible applications in quantum computation, where geometric phases are proposed to implement logic operations [37,26]. Geometric phases are believed to be robust against noise and control parameter fluctuations, due to their geometric nature. At the moment, this claim is just a conjecture, but it is supported by analytic results in some models of either adiabatic or non-adiabatic implementation. For a discussion on this point, see [39] and references therein.

The usual descriptions of geometric phases consider only a finite dimensional setting [26] or at least the Grassmann manifolds of finite dimensional subspaces of an infinite dimensional separable Hilbert space [11,12]. In this paper we study the arising of geometric phases without any adiabatic assumption and without any restriction on the dimensions of $\mathfrak{H}$ and K. Since it is natural to describe curves of subspaces by curves of projection operators, we study the time-dependent Hamiltonians which admit a curve of projections as an invariant and conditions to get geometric phases. The simple case of time-independent Hamiltonians which admit a closed geodesic as an invariant is investigated. For separable Hilbert spaces, the relative holonomies are characterized: they are reflections in K. More generally, we consider geodesic loops, i.e. loops composed by geodesic arcs, and investigate the subgroup of the holonomies generated by these special loops. We show that this group agrees with the entire $U(K)$ if K is finite dimensional or if $\dim(K) \le \dim(K^\perp)$. This obviously implies that the holonomy group of $\mathcal{A}$ agrees with $U(K)$. The remaining case, $\dim(K^\perp)$ finite and $\dim(K)$ infinite, is critical: we prove that the holonomy group is contained in the unitary Fredholm group of K and that, using holonomies relative to geodesic loops, one can approximate a generic unitary operator in the Fredholm group of K.

The plan of the paper is the following. In Section 2 we recall the topology and the structure of a holomorphic Banach manifold on $\mathrm{Gr}_K(\mathfrak{H})$ and the real analytic bundle structure on $\mathrm{St}(K, \mathfrak{H})$. A real analytic differential structure is directly given on the space $\mathcal{P}_K(\mathfrak{H})$ of orthogonal projections with range isomorphic to K, which makes the correspondence between subspaces and projections a diffeomorphism. In Section 3 the canonical connection $\mathcal{B}$ on the bundle $U(\mathfrak{H})$ over $\mathcal{P}_K(\mathfrak{H})$ is given. The induced connection $\mathcal{A}$ on the Stiefel bundle and the induced linear connection ∇ on $T\mathcal{P}_K(\mathfrak{H})$ are constructed. In Section 4 we introduce the geometric Hamiltonians, i.e. the Hamiltonians which admit a curve of projections as an invariant and whose dynamics on $\mathrm{St}(K, \mathfrak{H})$ implements the horizontal lift. In Section 5 we introduce phases and geometric phases and illustrate their meaning in quantum systems. In Section 6 we study the consistency of the group of the holonomies generated by closed geodesics and of the group of the holonomies generated by loops which are products of geodesic arcs. In the Appendix, details are given on the construction of the Hausdorff distance on the set $\mathrm{Gr}(\mathfrak{H})$ of the closed subspaces in $\mathfrak{H}$ and the equivalence with the opening distance is proved.

2. Manifolds of subspaces and manifolds of linear embeddings

2.1. *Topologies and distances on* $\mathrm{Gr}(\mathfrak{H})$ *and* $\mathcal{P}(\mathfrak{H})$

In the following we will assume that $\mathfrak{H}$ and K are complex separable Hilbert spaces. However, we stress that the statements given in Sections 2–5 still hold in the case of non-separable $\mathfrak{H}$ and K. We denote by $L(\mathfrak{H})$ the C^*-algebra of bounded operators of $\mathfrak{H}$, by $\mathcal{P}(\mathfrak{H})$ the set of orthogonal projections, endowed with the induced distance, given by

$$d(P, Q) := \|P - Q\| \quad \text{for } P, Q \in \mathcal{P}(\mathfrak{H});$$

the metric space $(\mathcal{P}(\mathfrak{H}), d)$ is complete and $d(P, Q) \leq 1$.

The action of the unitary group $U(\mathfrak{H})$ on $\mathcal{P}(\mathfrak{H})$, given by $P \mapsto UPU^\dagger$, is distance preserving. Moreover, P and Q belong to the same connected component of $\mathcal{P}(\mathfrak{H})$ if and only if they belong to the same unitary orbit or, equivalently, if the corresponding subspaces are isomorphic. For every Hilbert space K, we denote by $\mathcal{P}_K(\mathfrak{H})$ the connected component in $\mathcal{P}(\mathfrak{H})$ of projection operators on subspaces which are isomorphic to K.

The set of all (closed) subspaces of $\mathfrak{H}$ is denoted by $\mathrm{Gr}(\mathfrak{H})$. The subset of all subspaces of $\mathfrak{H}$ which are isomorphic to a given Hilbert space K will be denoted by $\mathrm{Gr}_K(\mathfrak{H})$. Now we will introduce a natural distance on $\mathrm{Gr}(\mathfrak{H})$. It is well known that the projective space $\mathbf{P}(\mathfrak{H}) = \mathrm{Gr}_{\mathbb{C}}(\mathfrak{H})$ is a complete metric space if endowed with the distance induced by the Fubini-Study metric [15,7]. Each non-zero subspace E of $\mathfrak{H}$ can be identified with the closed subset of $\mathbf{P}(\mathfrak{H})$ consisting of all 1-dimensional subspaces lying in E. So we can endow $\mathrm{Gr}(\mathfrak{H})$ with the distance D induced by the Hausdorff distance between closed subsets of $\mathbf{P}(\mathfrak{H})$. The details on this construction can be found in the Appendix.

Consider now the canonical bijection $\mathcal{P} : \mathrm{Gr}(\mathfrak{H}) \to \mathcal{P}(\mathfrak{H})$, $E \mapsto \mathcal{P}_E$ which associates to each subspace E the orthogonal projection on E. Let E and F be non-zero subspaces and let P and Q denote the orthogonal projections on E and F, respectively; then $\mathsf{D}(E, F) \leq \pi$ and

$$\|P - Q\| = \sin\left(\frac{1}{2}\mathsf{D}(E, F)\right)$$

as will be proved in the Appendix. By the above formula we see that the map $\mathcal{P} : \mathrm{Gr}(\mathfrak{H}) \to \mathcal{P}(\mathfrak{H})$ is a homeomorphism.

2.2. *The manifold* $\mathrm{Gr}_K(\mathfrak{H})$ *and related bundles*

In this paper we consider infinite dimensional holomorphic or real analytic Banach manifolds. For an introduction to these arguments we refer to [32]. The theory of infinite dimensional Banach manifolds is quite similar to the ordinary finite dimensional one. The more relevant differences arise from the fact that the closed subspaces of a Banach space are not necessarily complemented. We stress that every submanifold of a Banach manifold with model space E is modelled on a splitting subspace of E. For details, see [18].

In the following the term analytic will always mean real analytic and we will reserve the term holomorphic to the complex case. By homogeneous Banach analytic manifold we mean a Banach analytic manifold endowed with a transitive analytic action of a Banach Lie group. The notion of homogeneous Banach holomorphic manifold is analogously defined.

In this paper we are only interested in the actions of suitable subgroups of $GL(\mathfrak{H})$. As is well known, $GL(\mathfrak{H})$ is a complex Lie group with Lie algebra $L(\mathfrak{H})$. The unitary group $U(\mathfrak{H})$ is a closed real Lie subgroup of $GL(\mathfrak{H})$, with Lie algebra the real Banach Lie algebra $u(\mathfrak{H})$ of skew hermitian elements in $L(\mathfrak{H})$. For a introduction to infinite dimensional Banach Lie groups and homogeneous Banach manifolds, see [6,13,14,21,24] and [32].

Let K and $\mathfrak{H}$ be complex Hilbert spaces. Then $L(K, \mathfrak{H})$ denotes the complex Banach space of all bounded linear operators of K in $\mathfrak{H}$. A map $\varphi \in L(K, \mathfrak{H})$ is a (linear) *embedding* if it is a homeomorphism onto its image. Therefore the image $\mathrm{Im}(\varphi)$ is a subspace of $\mathfrak{H}$. We denote by $\mathrm{Emb}(K, \mathfrak{H})$ the set of the embeddings and by $\mathrm{St}(K, \mathfrak{H})$ the subset of the isometric embeddings. An embedding $u \in \mathrm{Emb}(K, \mathfrak{H})$ is isometric if and only if $u^\dagger u = \mathbf{1}_K$; therefore $\mathrm{St}(K, \mathfrak{H})$ is closed in $\mathrm{Emb}(K, \mathfrak{H})$. Moreover, $uu^\dagger$ is the projection operator on the range $\mathrm{Im}(u)$ of u.

For every embedding φ, $\varphi^\dagger\varphi \in GL(K)$. Conversely, let $\varphi \in L(K, \mathfrak{H})$ with $\varphi^\dagger\varphi \in GL(K)$. Then $\varphi \in \mathrm{Emb}(K, \mathfrak{H})$: denoting by $|\varphi|$ the unique positive operator in $GL(K)$ such that $|\varphi|^2 = \varphi^\dagger\varphi$ and considering the linear map $u_\varphi : K \to \mathfrak{H}$ defined by $u_\varphi := \varphi|\varphi|^{-1}$, one easily verifies that u_φ is an isometric embedding, with $\mathrm{Im}(u_\varphi) = \mathrm{Im}(\varphi)$. Therefore $\varphi = u_\varphi|\varphi|$ is an embedding. We conclude that

$$\mathrm{Emb}(K, \mathfrak{H}) = \{\varphi \in L(K, \mathfrak{H}) \ : \ \varphi^\dagger\varphi \in GL(K)\}.$$

Since $\mathrm{Emb}(K, \mathfrak{H})$ is open in $L(K, \mathfrak{H})$, it is a holomorphic manifold.

The maps $|\ |: \mathrm{Emb}(K,\mathfrak{H}) \to GL(\mathfrak{H})$, $\varphi \mapsto |\varphi|$ and $\flat : \mathrm{Emb}(K,\mathfrak{H}) \to \mathrm{Emb}(K,\mathfrak{H})$, $\varphi \mapsto \flat(\varphi) := u_\varphi$ are analytic. Therefore the map $\mathrm{Im} : \mathrm{Emb}(K,\mathfrak{H}) \to \mathrm{Gr}_K(\mathfrak{H}) \quad \varphi \mapsto \mathrm{Im}(\varphi)$ is continuous as the product of the continuous maps $\varphi \mapsto u_\varphi \mapsto u_\varphi u_\varphi^\dagger \mapsto \mathcal{P}^{-1}(u_\varphi u_\varphi^\dagger)$.

The group $GL(K)$ acts holomorphically and freely from the right on $\mathrm{Emb}(K,\mathfrak{H})$ by composition. Two elements of $\mathrm{Emb}(K,\mathfrak{H})$ lie in the same orbit if and only if they have the same image. By restriction, we obtain a distance preserving free right action of $U(K)$ on $\mathrm{St}(K,\mathfrak{H})$. Moreover, there is a left transitive action of $GL(\mathfrak{H})$ on $\mathrm{Emb}(K,\mathfrak{H})$, given by $(g,\varphi) \mapsto g\varphi$. The restriction to $U(\mathfrak{H})$ gives a left transitive action on $\mathrm{St}(K,\mathfrak{H})$. Analogously, a natural left transitive action $\tilde{\mu}$ of $GL(\mathfrak{H})$ on $\mathrm{Gr}_K(\mathfrak{H})$ is given by $\tilde{\mu}(g,E) := g.E$, with $g.E = \{gv, v \in E\}$. Its restriction μ to $U(\mathfrak{H})$ is transitive and distance preserving.

The holomorphic manifold structure on $\mathrm{Gr}_K(\mathfrak{H})$ is well known and has been more generally given for the set of splitting subspaces of a Banach space, see [9,14,32]. Here we summarize some relevant results.

Proposition 1. *The following statements hold.*

1. $\mathrm{Gr}_K(\mathfrak{H})$ *is a holomorphic Banach manifold. With respect to the left action* $\tilde{\mu}$ *of* $GL(\mathfrak{H})$, $\mathrm{Gr}_K(\mathfrak{H})$ *is a homogeneous holomorphic manifold. With respect to the left action* μ *of* $U(\mathfrak{H})$, $\mathrm{Gr}_K(\mathfrak{H})$ *is a homogeneous analytic manifold.*
2. *The map* $\mathrm{Im} : \mathrm{Emb}(K,\mathfrak{H}) \to \mathrm{Gr}_K(\mathfrak{H})$ *defines a holomorphic* $GL(K)$*-principal bundle and* $GL(\mathfrak{H})$ *acts as a group of holomorphic bundle isomorphisms.*
3. $\mathrm{St}(K,\mathfrak{H})$ *is a closed analytic submanifold of* $\mathrm{Emb}(K,\mathfrak{H})$. *The restriction of the projection* Im *to* $\mathrm{St}(K,\mathfrak{H})$ *defines an analytic* $U(K)$*-principal bundle, called the Stiefel bundle, on which* $U(\mathfrak{H})$ *acts transitively as a group of analytic bundle isomorphisms.*
4. *For every* $E \in \mathrm{Gr}_K(\mathfrak{H})$ *the tangent space* $T_E\mathrm{Gr}_K(\mathfrak{H})$ *is identified with* $L(E,E^\perp)$ *as a complex Banach space; the complex structure is given by* $J_E(z) := \mathrm{i}z$ *for* $z \in L(E,E^\perp)$ *and is norm preserving.*

Remark. 1. The manifold structure on $\mathrm{Gr}_K(\mathfrak{H})$ is compatible with the distance D. In fact a typical chart β_E at $E \in \mathrm{Gr}_K(\mathfrak{H})$ is constructed on $\mathcal{U}_E := \{F \in \mathrm{Gr}_K(\mathfrak{H}) : \mathsf{D}(E,F) < \pi\}$ as follows. For every $F \in \mathcal{U}_E$, there exists a unique $z_F \in L(E,E^\perp)$ such that $F = \mathrm{graph}(z_F) = \{x + z_F x \mid x \in E\}$ (see Corollary 2 in Appendix). The map $\beta_E : \mathcal{U}_E \to L(E,E^\perp)$, $\beta_E(F) = z_F$ results to be a homeomorphism with respect to the distance topologies. The inverse map is $\alpha_E : L(E,E^\perp) \to \mathcal{U}_E, \quad \alpha_E(z) = \mathrm{graph}(z)$.

2. If $\dim(K)$ is finite, $\mathrm{Gr}_K(\mathfrak{H})$ can be given a Kähler structure; if K is infinite dimensional, $\mathrm{Gr}_K(\mathfrak{H})$ can be given a Finsler structure.

One can also realize $\mathrm{Gr}_K(\mathfrak{H})$ as a coset space.

Proposition 2. *Let* K *be a subspace of* $\mathfrak{H}$, P *the orthogonal projection on* K *and* $\tilde{\mu}$ *the left action of* $GL(\mathfrak{H})$ *on* $\mathrm{Gr}_K(\mathfrak{H})$. *The following statements hold.*

1. *The isotropy group* $GL_K(\mathfrak{H})$ *at* K *is a Banach Lie subgroup with Lie algebra*

$$\mathfrak{k} := \{X \in L(\mathcal{H}) \mid P^\perp X P = 0\},$$

which is a complemented Lie subalgebra of $L(\mathfrak{H})$ *with complementary subspace*

$$\mathfrak{z} := \{Z \in L(\mathcal{H}) \mid Z = P^\perp Z P\}.$$

2. *The action* $\tilde{\mu}$ *is holomorphic and transitive. The orbit map* $\tilde{q}_K : GL(\mathfrak{H}) \to \mathrm{Gr}_K(\mathfrak{H})$, $\tilde{q}_K(g) := g.K$ *is a holomorphic submersion.*
3. *The coset space* $GL(\mathfrak{H})/GL_K(\mathfrak{H})$ *is a holomorphic Banach manifold. The map* $\tilde{q}_K$ *quotients to an equivariant biholomorphic diffeomorphism of* $GL(\mathfrak{H})/GL_K(\mathfrak{H})$ *with* $\mathrm{Gr}_K(\mathfrak{H})$.
4. *The map* $\tilde{q}_K : GL(\mathfrak{H}) \to \mathrm{Gr}_K(\mathfrak{H})$ *defines a holomorphic* $GL_K(\mathfrak{H})$*-principal bundle with holomorphic left action of* $GL(\mathfrak{H})$.

Proof. 1. It is obvious since $GL_K(\mathfrak{H})$ is the subgroup of the elements g in $GL(\mathfrak{H})$ which satisfy $P^\perp g P = 0$.

2. The action is transitive and holomorphic since it is the quotient action of the (transitive and) holomorphic action of $GL(\mathfrak{H})$ on $\mathrm{Emb}(K,\mathfrak{H})$ (see point 2 in Proposition 1). We verify that $\tilde{q}_K$ is a submersion. This is true since, for every $E \in \mathrm{Gr}_K(\mathfrak{H})$ the map $\sigma : \mathcal{U}_E \to GL(\mathfrak{H})$, $\sigma(F) := \mathbf{1}_{\mathfrak{H}} + z_F P_E$ is a local section of $\tilde{q}_K$.

3. One has just to apply Theorems 8.19 and 8.21 in [32] in their holomorphic versions.

4. It is an obvious consequence of the above statements. □

Proposition 3. *Let K and P be as in the above proposition and μ be the left action of $U(\mathfrak{H})$ on $\mathrm{Gr}_K(\mathfrak{H})$. The following statements hold.*

1. *The isotropy group at K is the Banach Lie subgroup $U_K(\mathfrak{H}) := U(K) \times U(K^\perp)$, with Lie algebra*

$$\mathfrak{k} := \mathfrak{u}(K) \oplus \mathfrak{u}(K^\perp),$$

which is a reductive Lie subalgebra of $u(\mathfrak{H})$ with complementary subspace the $Ad_{U_K(\mathfrak{H})}$-invariant subspace

$$\mathfrak{p} := \{W \in \mathfrak{u}(\mathfrak{H}) \mid WP = P^\perp W\}.$$

We denote by $\pi_{\mathfrak{k}}$ the projection on $\mathfrak{k}$ along $\mathfrak{p}$.

2. *The action μ is analytic and transitive. The orbit map $q_K : U(\mathfrak{H}) \to \mathrm{Gr}_K(\mathfrak{H})$, $q_K(U) = U.K$ is an analytic submersion.*

3. *The coset space $U(\mathfrak{H})/U_K(\mathfrak{H})$ is a symmetric analytic Banach manifold. The map q_K quotients to an equivariant bianalytic diffeomorphism of $U(\mathfrak{H})/U_K(\mathfrak{H})$ with $\mathrm{Gr}_K(\mathfrak{H})$.*

4. *The map $q_K : U(\mathfrak{H}) \to \mathrm{Gr}_K(\mathfrak{H})$ defines an analytic $U_K(\mathfrak{H})$-principal bundle with analytic left action of $U(\mathfrak{H})$.*

5. *The complex structure of $\mathrm{Gr}_K(\mathfrak{H})$ induces on $T\,(U(\mathfrak{H})/U_K(\mathfrak{H}))$ an invariant complex structure tensor $\mathcal{J}$ which is norm preserving. On $\mathfrak{p} \simeq T_o U(\mathfrak{H})/U_K(\mathfrak{H})$, $\mathcal{J}$ is given by $\mathcal{J}(W) = \mathrm{i}[W, P]$ and is $Ad_{U_K(\mathfrak{H})}$-invariant.*

Proof. 1. It is obvious.

2. To prove that μ is real analytic, we recall that $U(\mathfrak{H})$ is a Lie subgroup of $GL(\mathfrak{H})$, hence an analytic submanifold, and that μ is the restriction to $U(\mathfrak{H})$ of the natural action of $GL(\mathfrak{H})$ on $\mathrm{Gr}_K(\mathfrak{H})$. The only non-trivial point to check is that the orbit map $q_K : U(\mathfrak{H}) \to \mathrm{Gr}_K(\mathfrak{H})$ admits analytic local sections. Consider the chart $\beta_K : \mathcal{U}_K \to L(K, K^\perp)$ and compose it with the analytic map $z \mapsto V_z$ defined in Lemma 1 below. This gives the required local section for q_K.

3. One gives $U(\mathfrak{H})/U_K(\mathfrak{H})$ the structure of a Banach manifold as in Theorem 8.19 in [32]. For the definition of a symmetric manifold we refer to [15]. In this homogeneous space the symmetry at o is the quotient of adjoint action on $U(\mathfrak{H})$ of the unitary operator $2P - \mathbf{1}_{\mathfrak{H}}$. As usual, we denote by o the coset containing the unit operator. The second statement follows by Proposition 8.21 in [32].

4. It is an obvious consequence of the above statements.

5. By invariance, $\mathcal{J}$ is characterized by its restriction $\mathcal{J}|\mathfrak{p}$ to $\mathfrak{p} \simeq T_o(U(\mathfrak{H})/U_K(\mathfrak{H}))$. One can easily check that $(\mathcal{J}|\mathfrak{p})^2 = -\mathbf{1}_{\mathfrak{p}}$ and that the operator $\mathcal{J}|\mathfrak{p}$ is norm preserving and $Ad_{U_K(\mathfrak{H})}$-invariant. Now, $T_e q_K$ is the $\mathbb{R}$-linear isomorphism from $\mathfrak{p}$ to $T_K\mathrm{Gr}_K(\mathfrak{H}) \equiv L(K, K^\perp)$, given by the map $W \mapsto W|K$ for $W \in \mathfrak{p}$. Its inverse is the map $z \mapsto W_z := P^\perp z P - P z^\dagger P^\perp$. We have $\mathcal{J}(W_z)|K = i[W_z, P]|K = iz$, as one can easily verify. □

2.3. The manifold of the orthogonal projections

One could use Proposition 1 and the homeomorphism $\mathcal{P} : \mathrm{Gr}(\mathfrak{H}) \to \mathcal{P}(\mathfrak{H})$ to endow $\mathcal{P}(\mathfrak{H})$ with a holomorphic structure. Nevertheless, we find it interesting to give $\mathcal{P}_K(\mathfrak{H})$ an analytic manifold structure and an almost complex structure in an independent way and then prove that the canonical homeomorphism $\mathcal{P}$ results to be a bianalytic diffeomorphism preserving the almost complex structures. For simplicity, we assume that K is a non-zero subspace of $\mathfrak{H}$ and we denote by P the orthogonal projection on K.

Lemma 1. *For $z \in L(K, K^\perp)$, consider the operator $A_z := \mathbf{1}_{\mathfrak{H}} + P^\perp z P - P z^\dagger P^\perp$. The following statements hold.*

1. *$A_z \in GL(\mathfrak{H})$ and is normal, with polar decomposition $A_z = V_z|A_z|$ where V_z is unitary and $|A_z| \in GL(\mathfrak{H})$.*

2. $\mathrm{graph}(z) = V_z K = A_z K$ *and* $P_{\mathrm{graph}(z)} = A_z P A_z^{-1} = V_z P V_z^\dagger$.

Proof. 1. We have $A_z^\dagger = A_{-z}$. Therefore,

$$A_z A_z^\dagger = \mathbf{1}_{\mathfrak{H}} + P z^\dagger z P + P^\perp z z^\dagger P^\perp = A_{-z} A_z = A_z^\dagger A_z$$

so that A_z is a normal operator and $A_z^\dagger A_z = \rho_z^2 \oplus \rho_{z^\dagger}^2$ with

$$\rho_z = (\mathbf{1}_K + z^\dagger z)^{1/2} \quad \text{and} \quad \rho_{z^\dagger} = (\mathbf{1}_{K^\perp} + z z^\dagger)^{1/2}.$$

We remark that ρ_z is positive with $\mathbf{1}_E \le \rho_z \le (1 + \|z\|^2)^{1/2} \mathbf{1}_K$ and analogously for $\rho_{z^\dagger}$. Therefore, $\rho_z \in GL(K)$ and $\rho_{z^\dagger} \in GL(K^\perp)$. By $|A_z| \in GL(\mathfrak{H})$ we get $A_z \in GL(\mathfrak{H})$ and $V_z \in U(\mathfrak{H})$.

2. For every $v \in K$ we have $A_z v = v + zv$ so that graph$(z) = A_z(K) = V_z\rho_z(K) = V_z(K)$, since $\rho_z \in GL(K)$. We obtain that $P_{\mathrm{graph}(z)} = V_z P V_z^\dagger$ and it equals $A_z P A_z^{-1}$ since $|A_z|$ commutes with P. □

We define the map

$$\alpha_P : L(K, K^\perp) \to L(\mathfrak{H}), \qquad \alpha_P(z) = A_z P A_z^{-1}.$$

Lemma 2. *The map α_P is a homeomorphism with the open ball*

$$\mathcal{U}_P := \{Q \in \mathcal{P}(\mathfrak{H}) \mid \|Q - P\| < 1\}.$$

Its inverse is

$$\beta_P : \mathcal{U}_P \to L(K, K^\perp), \qquad \beta_P(Q) = P^\perp((Q|K)^{-1})^\dagger.$$

Proof. By Lemma 1, $\alpha_P(z)$ is the orthogonal projection operator on graph(z) so that $\alpha_P = \mathcal{P} \circ \alpha_K$ and it is a homeomorphism with $\mathcal{U}_P$ by Proposition 18 in the Appendix. The inverse of α_P is the continuous map $\beta_K \circ \mathcal{P}^{-1}$. We have to prove that it equals β_P. Let $Q \in \mathcal{U}_P$ and $F = \mathcal{P}^{-1}(Q)$. By applying the definition of β_K given in Corollary 2 in the Appendix, one has

$$\beta_P(Q) = P^\perp((Q|K)^{-1})^\dagger = P^\perp(P|F)^{-1} = \beta_K(F). \quad \square$$

Proposition 4. 1. *$\mathcal{P}_K(\mathfrak{H})$ is an analytic manifold and an analytic submanifold of $L_{sa}(\mathfrak{H})$, the real Banach space of selfadjoint elements in $L(\mathfrak{H})$.*

2. *The tangent space at P is*

$$\{A \in L_{s.a.}(\mathfrak{H}) \mid AP = P^\perp A\} =: \mathfrak{m}.$$

3. *A natural almost complex structure is defined by*

$$J_P(A) = \mathrm{i}[A, P] \quad \textit{for } A \in \mathfrak{m}.$$

In the identification of $\mathfrak{m}$ with $L(K, K^\perp)$ given by $A \mapsto A|K$, one has $J_P(A)|K = iA|K$ for every $A \in \mathfrak{m}$.

4. *The map $\mathcal{P} : \mathrm{Gr}_K(\mathfrak{H}) \to \mathcal{P}_K(\mathfrak{H})$ is an equivariant bianalytic diffeomorphism and preserves the almost complex structures. With respect to the action of $U(\mathfrak{H})$, $\mathcal{P}_K(\mathfrak{H})$ is a homogeneous analytic manifold.*

Proof. 1. Let $\tilde{P} \in \mathcal{P}_K(\mathfrak{H})$ and $V \in U(\mathfrak{H})$ be such that $\tilde{P} = VPV^\dagger$. We define

$$\beta_{\tilde{P}} : \mathcal{U}_{\tilde{P}} \to L(K, K^\perp), \qquad \beta_{\tilde{P}}(Q) = \beta_P(V^\dagger QV).$$

The chart change is

$$(\beta_{\tilde{P}} \circ \alpha_P)(z) = P^\perp((V^\dagger V_z P V_z^\dagger V|K)^{-1})^\dagger$$

which is analytic as a restriction of analytic maps. Therefore $\mathcal{P}_K(\mathfrak{H})$ is an analytic manifold. The inclusion of $\mathcal{P}_K(\mathfrak{H})$ in $L_{sa}(\mathfrak{H})$ is an analytic immersion since it locally looks as $\iota(z) = A_z P A_z^{-1}$, where $z \mapsto A_z$ is an affine map. Moreover, its derivative at 0, $\iota'(0) : L(K, K^\perp) \to L_{s.a.}(\mathfrak{H})$ is $\iota'(0)(z) = z \circ P + (z \circ P)^\dagger$, for $z \in L(K, K^\perp)$; hence it is injective and its image is a complemented subspace, as one can easily see.

2. For every C^1 curve $\wp$ in $\mathcal{P}(\mathfrak{H})$ the condition $\wp = \wp^2$ gives $\wp\dot{\wp} = \dot{\wp}\wp^\perp$.

3. It is obvious.

4. $\mathcal{P}$ is a bianalytic diffeomorphism since locally it agrees with $\alpha_P \circ \beta_K$ and its inverse is locally expressed by $\alpha_K \circ \beta_P$. By statement 3 and Proposition 3, it follows that $\mathcal{P}$ preserves the almost complex structures. Using the $U(\mathfrak{H})$-equivariance of $\mathcal{P}$, one obtains the remaining statements. □

A structure of analytic manifold can be introduced on the set of selfadjoint projections of a unital C^*-algebra $\mathfrak{A}$. This more general setting has been investigated in [2,8,27,28] and [36]. It was proved that the manifold of selfadjoint projections of $\mathfrak{A}$ is a (real) analytic submanifold of $\mathfrak{A}$ and admits an integrable complex structure.

3. The connection $\mathcal{B}$ and induced connections

In this section K will still denote a non-zero subspace of $\mathfrak{H}$ and P the orthogonal projection on K. Let θ be the canonical 1-form on $U(\mathfrak{H})$, the unique left invariant $u(\mathfrak{H})$-valued 1-form such that $\theta(X) = X$, for every $X \in \mathfrak{u}(\mathfrak{H})$. Since the subspace $\mathfrak{p}$ is $Ad_{U_K(\mathfrak{H})}$-invariant, the $\mathfrak{k}$-valued 1-form $\mathcal{B} := \pi_{\mathfrak{k}} \circ \theta$ is a connection 1-form on the $U_K(\mathfrak{H})$-principal bundle defined by $\pi_K : U(\mathfrak{H}) \to \mathcal{P}_K(\mathfrak{H})$, $\pi_K(U) = UPU^\dagger$. Moreover, left invariance of θ implies left invariance of $\mathcal{B}$.

We will identify the Stiefel bundle and the tangent bundle on $\mathcal{P}_K(\mathfrak{H})$ with bundles associated to the bundle $\pi_K : U(\mathfrak{H}) \to \mathcal{P}_K(\mathfrak{H})$ and we will construct on these bundles the connections induced by $\mathcal{B}$. The construction of bundles associated to principal bundles and induced connections, given in the setting of ordinary manifolds (see, e.g. [17]), can be extended without problems to Banach manifolds.

We consider the following analytic right action of $U_K(\mathfrak{H}) \equiv U(K) \times U(K^\perp)$ on $U(\mathfrak{H}) \times U(K)$,

$$r((U, S), (V_1 \times V_2)) := (U(V_1 \times V_2), V_1^\dagger S),$$

and endow the orbit space $U(\mathfrak{H}) \times_{U_K(\mathfrak{H})} U(K)$ with the unique analytic manifold structure such that the projection of $U(\mathfrak{H}) \times U(K)$ on the orbit space is a submersion. The induced projection of $U(\mathfrak{H}) \times_{U_K(\mathfrak{H})} U(K)$ onto $\mathcal{P}_K(\mathfrak{H})$ defines the associated bundle with fiber $U(K)$.

We can identify $U(\mathfrak{H}) \times_{U_K(\mathfrak{H})} U(K)$ with $\mathrm{St}(K, \mathfrak{H})$ as follows: define $\tilde{q_0} : U(\mathfrak{H}) \times U(K) \to \mathrm{St}(K, \mathfrak{H})$ by $\tilde{q_0}(U, S) := Uu_0S$, where $u_0 \in \mathrm{St}(K, \mathfrak{H})$ is the canonical inclusion. The map $\tilde{q_0}$ quotients to an isomorphism of $U(\mathfrak{H}) \times_{U_K(\mathfrak{H})} U(K)$ with $\mathrm{St}(K, \mathfrak{H})$. We remark also that the map $U \mapsto Uu_0$ quotients to a diffeomorphism of $\mathrm{St}(K, \mathfrak{H})$ with $U(\mathfrak{H})/U(K^\perp)$.

The principal connection $\mathcal{B}$ induces a left invariant connection $\mathcal{A}$ on $\mathrm{St}(K, \mathfrak{H})$. For every C^1 curve $\gamma : \mathcal{I} \to \mathcal{P}_K(\mathfrak{H})$, where $\mathcal{I}$ is an interval of the real line, and $t, t_0 \in \mathcal{I}$,

$$\mathsf{Pt}^{\mathcal{A}}(\gamma, t, \tilde{q_0}(U, S)) = \tilde{q_0}(\mathsf{Pt}^{\mathcal{B}}(\gamma, t, U), S) \tag{3.1}$$

for every $U \in U(\mathfrak{H})$ over $\gamma(t_0)$ and $S \in U(K)$. Here $\mathsf{Pt}^{\mathcal{A}}$ and $\mathsf{Pt}^{\mathcal{B}}$ denote the parallel transport w.r.t. $\mathcal{A}$ and $\mathcal{B}$, respectively. The connection $\mathcal{A}$ results to be a principal connection on the Stiefel bundle given by $\pi : \mathrm{St}(K, \mathfrak{H}) \to \mathcal{P}_K(\mathfrak{H})$, $\pi(u) = u^\dagger u$, as is illustrated in the following proposition.

Proposition 5. 1. *The tangent space at u of* $\mathrm{St}(K, \mathfrak{H})$ *is the space of all $\xi \in L(K, \mathfrak{H})$ such that*

$$\xi^\dagger u + u^\dagger \xi = 0_K. \tag{3.2}$$

2. *The vertical subspace at u is the subspace of $T_u St(K, \mathfrak{H})$ consisting of all $\xi \in L(K, \mathfrak{H})$ such that* $\mathrm{Im}(\xi) \subset \mathrm{Im}(u)$.

3. *The $u(K)$-valued 1-form $u^\dagger \mathrm{d}u$ defined by*

$$u^\dagger \mathrm{d}u(\xi) = u^\dagger \xi, \qquad \xi \in T_u\mathrm{St}(K, \mathfrak{H})$$

is a principal connection 1-form on $\mathrm{St}(K, \mathfrak{H})$ *whose parallel transport is* $\mathsf{Pt}^{\mathcal{A}}$. *Its horizontal subspace* $H_u(\mathrm{St}(K, \mathfrak{H}))$ *at u consists of all $\xi \in T_u\mathrm{St}(K, \mathfrak{H})$ such that* $\mathrm{Im}(\xi) \subset \mathrm{Im}(u)^\perp$.

Proof. 1. Differentiating the condition $u^\dagger u = \mathbf{1}_K$, we obtain formula (3.2).

2. Differentiating the mapping $\pi(u) = uu^\dagger$, we get

$$(T_u\pi)(\xi) = \xi u^\dagger + u\xi^\dagger \in L(\mathfrak{H}).$$

The vertical subspace at u of $\mathrm{St}(K, \mathfrak{H})$ is the kernel of $T_u\pi$, described by the linear equations

$$\xi^\dagger u + u^\dagger \xi = 0_K \quad \text{and} \quad \xi u^\dagger + u\xi^\dagger = 0_{\mathfrak{H}}.$$

Therefore $\xi^\dagger u = -u^\dagger \xi$ and $\xi + u\xi^\dagger u = 0_K$, so that $\xi = uu^\dagger \xi$ as required.

3. One can easily verify that $u^\dagger du$ is a principal connection 1-form on $\mathrm{St}(K, \mathfrak{H})$ and that its kernel at u is $H_u(\mathrm{St}(K, \mathfrak{H}))$. Let u_0, U, S, γ be as in formula (3.1). Denote $u(t) := \mathsf{Pt}^{\mathcal{A}}(\gamma, t, Uu_0S)$ and $U(t) := \mathsf{Pt}^{\mathcal{B}}(\gamma, t, U)$. The formula (3.1) can be written simply as $u(t) = U(t)u_0S$. Differentiating, we obtain

$$u^\dagger(t)\dot{u}(t) = S^\dagger u_0^\dagger U^\dagger(t)\dot{U}(t)u_0S = 0_K$$

since $U^\dagger(t)\dot{U}(t) \in \mathfrak{p}$. $\square$

For $\mathfrak{H}$ finite dimensional, the connection $\mathcal{A}$ was proved to be a universal connection for $U(K)$-principal bundles in the celebrated paper of Narasimhan and Ramanan [23].

For G a finite dimensional Lie group and H a closed reductive connected subgroup, the theory of G-invariant linear connections on $T(G/H)$ is well-established (see e.g. Theorem 3.2.31 in [19]). Identifying $\mathcal{P}_K(\mathfrak{H})$ with $U(\mathfrak{H})/U_K(\mathfrak{H})$ we can analogously construct a linear connection on $T\mathcal{P}_K(\mathfrak{H})$ induced by $\mathcal{B}$.

Since $\mathfrak{m}$ is $Ad_{U_K(\mathfrak{H})}$-invariant, we can consider the joint right action of $U_K(\mathfrak{H})$ on $U(\mathfrak{H}) \times \mathfrak{m}$ given by

$$(U, A).S = (US, S^\dagger AS), \quad S \in U_K(\mathfrak{H}).$$

We denote by $U(\mathfrak{H}) \times_{U_K(\mathfrak{H})} \mathfrak{m}$ the orbit space and by $\tilde{q}$ the projection of $U(\mathfrak{H}) \times \mathfrak{m}$ on $T\mathcal{P}_K(\mathfrak{H})$ given by $\tilde{q}(U, A) \mapsto UAU^\dagger$, which quotients to a diffeomorphism of $U(\mathfrak{H}) \times_{U_K(\mathfrak{H})} \mathfrak{m}$ with $T\mathcal{P}_K(\mathfrak{H})$, so that one can identify the tangent bundle on $\mathcal{P}_K(\mathfrak{H})$ with the associated vector bundle of fiber $\mathfrak{m}$. Moreover, the invariant principal connection $\mathcal{B}$ induces an invariant linear connection ∇ on $U(\mathfrak{H}) \times_{U_K(\mathfrak{H})} \mathfrak{m}$. Again, for every C^1 curve $\wp : \mathcal{I} \to \mathcal{P}_K(\mathfrak{H})$, $t, t_0 \in \mathcal{I}$, and for every $U \in U(\mathfrak{H})$ over $\wp(t_0)$ and $A \in \mathfrak{m}$, one has

$$\mathsf{Pt}^\nabla(\wp, t, \tilde{q}(U, A)) = \tilde{q}(\mathsf{Pt}^\mathcal{B}(\wp, t, U), A), \tag{3.3}$$

where Pt^∇ is the parallel transport w.r.t. the associated linear connection ∇.

Proposition 6. *For $Y \in \mathfrak{m}$ consider the vector field Y^* defined by $Y^*(Q) := -\mathrm{i}[Y, Q]$ for $Q \in \mathcal{P}_K(\mathfrak{H})$. The linear connection ∇ on $T\mathcal{P}_K(\mathfrak{H})$ associated to $\mathcal{B}$ is uniquely defined by*

$$\nabla_{Y^*(P)} X = [Y^*, X](P)$$

for all local vector fields X around P, where $[\ ,\ ]$ denotes the Lie bracket of vector fields.

Proof. The field $Y^\star$ is complete, with flow given by $(Q, t) \mapsto \mathrm{e}^{-\mathrm{i}Yt} Q \mathrm{e}^{\mathrm{i}Yt}$. Its integral curve at P is the curve $\wp(t) := \mathrm{e}^{-\mathrm{i}Yt} P \mathrm{e}^{\mathrm{i}Yt}$. We remark that $U(t) = \mathrm{e}^{\mathrm{i}Yt}$ is $\mathcal{B}$-horizontal for $Y \in \mathfrak{m}$, since $\mathcal{B}(\dot{U}(t)) = \mathrm{i}(PYP + P^\perp Y P^\perp) = 0$. Formula (3.3) gives

$$\mathsf{Pt}^\nabla(\wp, t, A) = \mathrm{e}^{-\mathrm{i}Yt} A \mathrm{e}^{\mathrm{i}Yt} \quad \forall A \in \mathfrak{m}. \tag{3.4}$$

Let X be a local vector field around P. Then

$$\begin{aligned}\nabla_{Y^*(P)} X &= \lim_{t \to 0} \frac{1}{t} \left(\mathsf{Pt}^\nabla(\wp^{-1}, t, X(\wp(t))) - X(\wp(0)) \right) \\ &= \lim_{t \to 0} \frac{1}{t} \left(\mathrm{e}^{\mathrm{i}Yt} X(\wp(t)) \mathrm{e}^{-\mathrm{i}Yt} - X(\wp(0)) \right) = [Y^*, X](P).\end{aligned}$$

By invariance, this defines ∇ completely. □

The following proposition illustrates the geodesics in $\mathcal{P}_K(\mathfrak{H})$.

Proposition 7. *Let K, P and $\mathfrak{p}$ be as in Proposition 3 and $W \in \mathfrak{p}$. The following statements hold.*

1. *The curve of unitary operators $U(t) = \mathrm{e}^{Wt}$, $t \in \mathbb{R}$, is $\mathcal{B}$-horizontal.*
2. *The curve of isometric embeddings $u(t) = \mathrm{e}^{Wt} u$, $t \in \mathbb{R}$, is $\mathcal{A}$-horizontal for every $u \in \mathrm{St}(K, \mathfrak{H})$ with $uu^\dagger = P$.*
3. *The curve of projection operators $\wp(t) = \mathrm{e}^{tW} P \mathrm{e}^{-tW}$, $t \in \mathbb{R}$, is a geodesic in $\mathcal{P}_K(\mathfrak{H})$ and every geodesic in $\mathcal{P}_K(\mathfrak{H})$ starting from P can be represented in this form.*

Proof. 1. See the proof of point 2 in Proposition 6.

2. This follows by Eq. (3.1).

3. For the proof, see Theorem XI. 3.2, Vol. II in [15]. □

As an obvious consequence of the above proposition, the geodesics in $\mathrm{Gr}_K(\mathfrak{H})$ starting from K are of the form $\mathrm{e}^{Wt}.K$ with $W \in \mathfrak{p}$.

4. Curves of projections and geometric Hamiltonians

Now we discuss "dynamically driven" lifts in $\mathrm{St}(K,\mathfrak{H})$ of a curve in the space of projections, i.e. lifts generated by the evolution governed by a time dependent Hamiltonian. We give the conditions on the Hamiltonian which ensure that these lifts are horizontal.

Let $C:\mathcal{I}\to L(\mathfrak{H})$ be a continuous curve of operators. In this section we assume that $0\in\mathcal{I}$ and we refer to $C(0)$ as the starting point of C. We denote by $\dot{C}$ the curve of derivatives, when C is a C^1 curve. For a pair C_1, C_2 of curves we denote by C_1C_2 the curve $(C_1C_2)(t):=C_1(t)C_2(t)$. We also set $[C_1,C_2]:=C_1C_2-C_2C_1$. We denote a C^1 curve in $\mathcal{P}_K(\mathfrak{H})$ by $\wp$, its starting point $\wp(0)$ by P and the range of P by K. From now on, U will mean a C^1 curve in $U(\mathfrak{H})$ with $U(0)=\mathbf{1}_{\mathfrak{H}}$ and u will mean a C^1 curve in $\mathrm{St}(K,\mathfrak{H})$. We denote by u_0 the starting point $u(0)$.

Let Q be any orthogonal projection in $\mathfrak{H}$. We will say that $X\in L(\mathfrak{H})$ is Q-diagonal if $QXQ^\perp=0$ and $Q^\perp XQ=0$; we will say that X is Q-off-diagonal if $QXQ=0$ and $Q^\perp XQ^\perp=0$.

Let H be a time-dependent Hamiltonian, i.e. a continuous curve of bounded skew-adjoint operators in $\mathfrak{H}$. We will say that the Hamiltonian H is $\wp$-diagonal (or $\wp$-off-diagonal) if $H(t)$ is $\wp(t)$-diagonal (or $\wp(t)$-off-diagonal) for all t. If $\wp$ is a geodesic with $\wp(t)=\mathrm{e}^{tW}P\mathrm{e}^{-tW}$ as in Proposition 7, then W is $\wp$-off-diagonal, as one can easily verify. Given $\wp$, one can construct a $\wp$-off-diagonal Hamiltonian: actually, the time-dependent Hamiltonian

$$H^\wp:=[\dot{\wp},\wp]=\wp^\perp\dot{\wp}\wp-\wp\dot{\wp}\wp^\perp$$

is $\wp$-off-diagonal.

Let a curve $\wp$ and a Hamiltonian H satisfy

$$\dot{\wp}=[H,\wp]. \tag{4.1}$$

In this case we say that $\wp$ is an invariant of H or that H admits $\wp$ as an invariant. This notion can be extended in an obvious way to continuous piecewise C^1 curves.

The following proposition characterizes the Hamiltonians which admit a curve $\wp$ as an invariant.

Proposition 8. *Let $\wp$ be a C^1 curve in $\mathcal{P}_K(\mathfrak{H})$. A Hamiltonian H admits $\wp$ as an invariant if and only if*

$$H=H^\wp+H^\delta$$

where H^δ is $\wp$-diagonal.

Proof. Assuming Eq. (4.1), we have $H\wp=\dot{\wp}+\wp H$, so that $\wp^\perp H\wp=\wp^\perp\dot{\wp}\wp$ and $\wp H\wp^\perp=-\wp\dot{\wp}\wp^\perp$. Conversely, we observe that $H^\wp$ satisfies Eq. (4.1) and that $[\wp,H^\delta]=0$. □

We remark that $\wp$ is a geodesic if and only if $H^\wp$ is constant. Assuming $W=H^\wp(t)$ for every t, then W is $\wp(0)$-off-diagonal and $\wp(t)=\mathrm{e}^{Wt}\wp(0)\mathrm{e}^{-Wt}$. Proposition 7 assures that $\wp$ is a geodesic starting from $\wp(0)$. The converse is immediate.

Let $U(t)$ be the C^1 curve in $U(\mathfrak{H})$ satisfying the Schrödinger equation defined by H:

$$\dot{U}(t)U^\dagger(t)=H(t),\qquad U(0)=\mathbf{1}_{\mathfrak{H}}. \tag{4.2}$$

For every $u_0\in\mathrm{St}(K,\mathfrak{H})$, the curve $u(t):=U(t)u_0$ in $\mathrm{St}(K,\mathfrak{H})$ satisfies

$$\dot{u}(t)=H(t)u(t),\qquad u(0)=u_0. \tag{4.3}$$

Proposition 9. *Let H be a Hamiltonian which admits $\wp$ as an invariant and let u be a curve as in Eq. (4.3), with u_0 over P. The following statements hold.*

1. *The curve u is a lift of $\wp$ in* $\mathrm{St}(K,\mathfrak{H})$ *starting from u_0.*
2. *The curve u is the horizontal lift of $\wp$ w.r.t. the connection $\mathcal{A}$ on* $\mathrm{St}(K,\mathfrak{H})$ *if and only if $\wp H\wp=0$.*

Proof. 1. One can easily verify that $u(t)u(t)^\dagger=U(t)PU(t)^\dagger$ is the (unique) solution of Eq. (4.1) with initial condition P.

2. Let $\wp H\wp=0$. Since $u(t)$ take values in the range of $\wp(t)$, Eq. (4.3) becomes

$$\dot{u}(t)=H(t)\wp(t)u(t)=\wp^\perp(t)\dot{\wp}(t)\wp(t)u(t). \tag{4.4}$$

Hence $\mathcal{A}(\dot{u}(t))=u^\dagger(t)(\wp^\perp(t)\dot{\wp}(t)\wp(t))u(t)=0$, since $u=\wp u$ implies $u^\dagger\wp^\perp=0$.

Conversely, assume that $\mathcal{A}(\dot{u}) = u^\dagger \dot{u} = 0$. By composing the equation $u^\dagger \dot{u} = 0$ with u on the left and with $u^\dagger$ on the right, and by using the equation $\dot{u} = Hu$, we obtain $\wp \dot{u} u^\dagger = \wp H u u^\dagger = \wp H \wp = 0$. □

We remark that every lift u of a curve $\wp$ is piecewise a solution of Eq. (4.3) for some Hamiltonian H. This follows from the fact that $\mathrm{St}(K, \mathfrak{H})$ is isomorphic to the homogeneous space $U(\mathfrak{H})/U(K^\perp)$ and that the projection $U(\mathfrak{H}) \to \mathrm{St}(K, \mathfrak{H})$ admits local sections.

Proposition 9 suggests the following terminology: a Hamiltonian H which admits $\wp$ as an invariant will be called *geometric* for $\wp$ if $\wp H \wp = 0$ is satisfied. If one reparametrizes $\wp$ by a diffeomorphism $\tau \to t(\tau)$ of the interval $\mathcal{I}$, the curve $\tilde{\wp}(\tau) := \wp(t(\tau))$ is an invariant for the Hamiltonian

$$\tilde{H}(\tau) := H(t(\tau)) \frac{\mathrm{d}t}{\mathrm{d}\tau}(\tau). \tag{4.5}$$

Moreover, if H is geometric for $\wp$, then $\tilde{H}$ is geometric for $\tilde{\wp}$.

Corollary 1. *Let $\wp$ be a C^1 curve in $\mathcal{P}_K(\mathfrak{H})$ with $\wp(0) = P$ and let V be the C^1 curve in $U(\mathfrak{H})$ satisfying*

$$\dot{V}(t) V^\dagger(t) = H^\wp(t) \qquad V(0) = \mathbf{1}_\mathfrak{H}.$$

The following statements hold.

1. *V is the $\mathcal{B}$-horizontal lift in $U(\mathfrak{H})$ of $\wp$ starting on $\mathbf{1}_\mathfrak{H}$.*
2. *$H^\wp$ is the unique Hamiltonian which is geometric both in* $\mathrm{St}(K, \mathfrak{H})$ *for $\wp$ and in* $\mathrm{St}(K^\perp, \mathfrak{H})$ *for $\wp^\perp$.*

Proof. 1. The Hamiltonian $H^\wp$ admits the curve $\wp$ as an invariant, thus $\wp(t) = V(t) P V^\dagger(t)$, so that V is a lift of $\wp$. We will show that $\mathcal{B}(\dot{V}) = P(V^\dagger H^\wp V)P + P^\perp (V^\dagger H^\wp V) P^\perp$ is zero. The first term $P(V^\dagger H^\wp V)P$ is zero: by composing it with V on the left and with $V^\dagger$ on the right, we get $\wp H^\wp \wp = 0$. The proof that $P^\perp (V^\dagger H^\wp V) P^\perp = 0$ is analogous.

2. It follows from point 2 of Proposition 9. □

Now we get in our framework a well known result (see for instance [22]).

Proposition 10. *Let us denote by U the solution of Eq. (4.2) for $H = H^\wp + H^\delta$. The following statements hold.*

1. *$U(t) = V(t) U_I(t)$, where $U_I(t)$ is the solution of Eq. (4.2) for the Hamiltonian $H_I = V^\dagger H^\delta V$.*
2. *H_I is P-diagonal.*
3. *U_I takes values in $U(K) \times U(K^\perp)$. Its components $U_1 \in U(K)$ and $U_2 \in U(K^\perp)$ satisfy the equations $\dot{U}_1 = H_1 U_1$ and $\dot{U}_2 = H_2 U_2$, where H_1 and H_2 are the restrictions of H_I to K and $K^\perp$, respectively.*

Proof. 1. Taking the derivative of U_I, we obtain

$$\dot{U}_I = -V^\dagger H^\wp U + V^\dagger (H^\wp + H^\delta) U = V^\dagger H^\delta V U_I.$$

2. From $V(t)P = \wp(t) V(t)$ and $P V^\dagger(t) = V^\dagger(t) \wp(t)$, we get $P V^\dagger(t) H^\delta(t) V(t) P^\perp = V^\dagger(t) \wp(t) H^\delta(t) \wp^\perp(t) V(t) = 0$, since H^δ is $\wp$-diagonal.

3. From $\wp(t) = V(t) P V^\dagger(t) = U(t) P U^\dagger(t)$, we get $P = U_I(t) P U_I(t)$. The remaining of this statement follows from the uniqueness of the solution of Eq. (4.2). □

Using the formalism of the above proposition, the evolution of any $u_0 \in \mathrm{St}(K, \mathfrak{H})$ with $u_0 u_0^\dagger = P$ is $u(t) = V(t) U_I(t) u_0$. We are interested in Hamiltonians for which this evolution differs from the horizontal lift in a simple way.

Example 1. $\wp H^\delta \wp = 0$. This is the case of Hamiltonians which are geometric for $\wp$, so that $u(t)$ agrees with the horizontal lift of $\wp$, as proved in Proposition 9.

Example 2. $(\wp H^\delta \wp)(t) = \mathrm{i}\lambda(t) \wp(t)$ with $\lambda(t) \in \mathbb{R}$. Then $U_I(t) = \mathrm{e}^{\mathrm{i}\alpha(t)} P + U_2(t)$ where $\alpha(t) = \int_0^t \lambda(s)\mathrm{d}s$, and $U_2(t) \in U(K^\perp)$, so that $u(t) = \mathrm{e}^{\mathrm{i}\alpha(t)} V(t) u_0$. In this case u differs from the horizontal lift only by a numerical factor which is assumed under control. If $\wp$ is a curve of monodimensional projections the condition is obviously satisfied.

The second example suggests the interest of the case in which the factors in $U = VU_I$ are commuting. We restrict our attention to a time-independent Hamiltonian $H(t) = X$ and assume that

$$[X_{\mathfrak{k}}, X_{\mathfrak{p}}] = 0 \tag{4.6}$$

where $X_{\mathfrak{k}}$ and $X_{\mathfrak{p}}$ are the projections on $\mathfrak{k}$ and $\mathfrak{p}$, respectively. The following proposition holds.

Proposition 11. *Let $X \in \mathfrak{u}(\mathcal{H})$ and $\wp(t) = \mathrm{e}^{tX} P e^{-tX}$. Then X satisfies Eq. (4.6) if and only if $\wp$ is a geodesic.*

Proof. Assuming Eq. (4.6), we have $\wp(t) = \mathrm{e}^{tX} P\mathrm{e}^{-tX} = e^{tX_{\mathfrak{p}}} e^{tX_{\mathfrak{k}}} P e^{-tX_{\mathfrak{k}}} e^{-tX_{\mathfrak{p}}} = e^{tX_{\mathfrak{p}}} P e^{-tX_{\mathfrak{p}}}$. Thus $\wp$ is a geodesic by Proposition 7.

Conversely, let $\wp(t) = \mathrm{e}^{tX} P e^{-tX} = \mathrm{e}^{tW} P \mathrm{e}^{-tW}$, with $W \in \mathfrak{p}$. From $\dot{\wp}(0) = [X, P] = [W, P]$ it follows easily that $W = X_{\mathfrak{p}}$. Moreover $U_I(t) = \mathrm{e}^{-tX_{\mathfrak{p}}} \mathrm{e}^{tX}$ is a curve in $U_K(\mathfrak{H})$ whose generator $(\dot{U}_I U_I^{\dagger})(t) = \mathrm{e}^{-tX_{\mathfrak{k}}}(X - X_{\mathfrak{k}})\mathrm{e}^{tX_{\mathfrak{k}}} = \mathrm{e}^{-tX_{\mathfrak{k}}} X_{\mathfrak{p}} \mathrm{e}^{tX_{\mathfrak{p}}}$ belongs to $\mathfrak{k}$ for every t. This implies that $[X_{\mathfrak{p}}, X_{\mathfrak{k}}]$ belongs to $\mathfrak{k}$. Taking into account that $\mathfrak{p}$ is $ad_{\mathfrak{k}}$-invariant, we obtain that $[X_{\mathfrak{p}}, X_{\mathfrak{k}}] = 0$. □

5. Geometric phases

Assume that $\wp$ is a loop at P, i.e. that $\mathcal{I} = [0, T]$ and that $\wp(T) = \wp(0) = P$. Let $u(t)$ be a C^1 lift of $\wp$ in $\mathrm{St}(K, \mathfrak{H})$ starting from u_0 over P. There exists a unique $\Phi \in U(K)$ such that $u(T) = u_0 \Phi$. We call Φ the phase acquired by u_0 along the lift u or the phase of the lift u. We say that the phase Φ is a *geometric phase* if $\Phi = \mathrm{Hol}(\mathcal{A}, \wp, u_0)$. The above definitions can be given analogously when the curves $\wp$ and u are piecewise C^1.

It could appear inappropriate to call "phase" a unitary operator. In the case $K \simeq \mathbb{C}$, the unitary operators are of the form $\mathrm{e}^{\mathrm{i}\theta}$, $\theta \in \mathbb{R} \pmod{2\pi}$ where θ is the phase. Moreover, for $\dim(K) > 1$ the term non-Abelian geometric phase is commonly used to denote the above defined unitary operator. Here we adopt a general term valid for any dimension.

If a Hamiltonian $H(t)$ admits a loop $\wp$ as an invariant and $U(t)$ is the solution of Eq. (4.2), the curve $u(t) = U(t)u_0$ is a lift of $\wp$ and its phase satisfies

$$U(T)u_0 = u_0 \Phi. \tag{5.1}$$

If K is a subspace of $\mathfrak{H}$ and u_0 is the canonical inclusion, then Φ agrees with the restriction of $U(T)$ to K.

Many different Hamiltonians can give the same phase. Indeed, let $X(t)$ be a C^1 curve in $U(\mathfrak{H})$ starting from $\mathbf{1}_{\mathfrak{H}}$ such that $X(T)u_0 = u_0$, then the Hamiltonian $\dot{W}(t)W^{\dagger}(t)$ with $W(t) = U(t)X(t)$ gives the same phase as H. If a Hamiltonian H gives a geometric phase at u_0, the same H induces a geometric phase at every $\tilde{u}_0$ over P: let $\tilde{u}_0 = u_0 S$, with $S \in U(K)$. Then

$$\tilde{\Phi} = \tilde{u}_0^{\dagger} U(T) \tilde{u}_0 = S^{-1} \Phi S = \mathrm{Hol}(\mathcal{A}, \wp, \tilde{u}_0).$$

The phase does not change under reparametrization of $\wp$ and H as in Eq. (4.5).

The above geometrical setting can be applied to very different physical contexts. Let $\mathfrak{H}$ be the Hilbert space associated to the description of a quantum system. The set of (pure) states is represented by $\mathrm{Gr}_{\mathbb{C}}(\mathfrak{H}) = \mathbf{P}(\mathfrak{H})$, the projective space of $\mathfrak{H}$, and $\mathrm{St}(\mathbb{C}, \mathfrak{H})$ can be identified with $S(\mathfrak{H})$, the set of unit vectors of $\mathfrak{H}$. Of course, these vectors do not have a direct physical meaning: $\psi \in S(\mathfrak{H})$ is just a representative for the corresponding state $\hat{\psi}$, the 1-dimensional subspace spanned by ψ. The dynamics of the system is governed by a suitable Hamiltonian $H(t)$ according to the Schrödinger equation $\dot{\psi}(t) = H(t)\psi(t)$. We denote by $U(t)$ the corresponding evolution operator.

Let us suppose that a state $\hat{\psi}$ evolves along a loop, i.e. that $\hat{\psi}(T) = \hat{\psi}(0)$. Since $\psi(t) = U(t)\psi(0)$ is a lift of the curve $\hat{\psi}(t)$, then $\psi(T) = \mathrm{e}^{\mathrm{i}\theta}\psi(0)$, with $\theta \in [0, 2\pi]$, where $\mathrm{e}^{\mathrm{i}\theta}$ is the phase acquired by $\psi(0)$ along $\psi(t)$. This phase is not detectable by experiments on the state $\hat{\psi}$ alone, but it is measurable by interference experiments in which superpositions of $\hat{\psi}$ with another fiducial state are observed [10,31]. Let $\wp(t)$ be the orthogonal projection on $\hat{\psi}(t)$. Since $\wp$ is a loop of monodimensional projections we are in the case of Example 2, so that $\psi(T) = \mathrm{e}^{\mathrm{i}\alpha(T)}\mathrm{e}^{\mathrm{i}\varphi}\psi(0)$ where $\mathrm{e}^{\mathrm{i}\varphi} = \mathrm{Hol}(\mathcal{A}, \wp, \psi_0)$ is the geometric phase and the first term (or, more appropriately, $\alpha(T)$) is often called the dynamic phase. As $U(1)$ is Abelian, the geometric phase does not depend on the starting point ψ_0. This factorization of the phase was firstly pointed out by Berry [3]; the geometric phase is referred to as Berry's phase.

Things go in a different way when $E \in \mathrm{Gr}_K(\mathfrak{H})$ with $\dim(K) > 1$ is considered. The subspace E can be identified with the corresponding subset of states $\hat{E} := \{\hat{\psi}, \psi \in S(E)\}$. The elements of $\mathrm{St}(K, \mathfrak{H})$ do not have a direct physical meaning. Let $U(t)$ be the evolution operator for a Hamiltonian H. The curve $E(t) = U(t).E$ in $\mathrm{Gr}_K(\mathfrak{H})$ describes the evolution of the corresponding set of states $\hat{E}(t)$. This gives a physical meaning to the fact that a curve $\wp$ of projection operators is an invariant for H. Of course, $E(T) = E$ does not mean that every state in $\hat{E}$ undergoes a cyclic evolution: one has just $\hat{\psi}(T) = \widehat{U(T)\psi} \in \hat{E}$ for every $\psi \in E$; therefore the restriction of $U(T)$ to E is a physically detectable transformation. If an element $u_0 \in \mathrm{St}(K, \mathfrak{H})$ over E is chosen, the restriction of $U(T)$ to E is related with the phase Φ acquired by u_0 along $u(t) = U(t)u_0$. Eq. (5.1) gives $\Phi = u_0^\dagger U(T)u_0$, so that the phase is simply the representation in $U(K)$ of the restriction of $U(T)$ to E by means of the embedding u_0. Of course, if u_0 is changed, the phase changes by conjugation. Assuming that K is a subspace of $\mathfrak{H}$ and using the notation in Proposition 10, we have

$$\Phi = u_0^\dagger U(T)u_0 = u_0^\dagger V(T)U_I(T)u_0 = \mathrm{Hol}(\mathcal{A}, \wp, u_0)u_0^\dagger U_I(T)u_0,$$

where $u_0^\dagger U_I(T)u_0$ is the non-Abelian analogous of the dynamic phase.

The problem arises to find a reasonable protocol to implement a general transformation in E by varying Hamiltonians and loops. By taking every loop $\wp$ with base point P in $\mathcal{P}_K(\mathfrak{H})$ and choosing $H^\wp$ as Hamiltonian, one obtains every element of the holonomy group. It is well known that, if $\dim(\mathfrak{H})$ is finite, the holonomy group is the entire group $U(K)$, so that every unitary transformation in E can, in principle, be implemented in this way. However, many different loops give the same holonomy, so it is expected and desirable to obtain the entire holonomy group considering only a selected family of loops. We call a *geodesic arc* in $\mathcal{P}_K(\mathfrak{H})$ the restriction of a geodesic to a compact interval. In the following section we investigate the consistence of the holonomies generated by these special loops.

A standard example of a physical system in which Berry's phases arise is a spin $\frac{1}{2}$ particle in a magnetic field, usually assumed time-dependent. Suppose instead that $H = \mathrm{i}\frac{\mu}{2}B\sigma_z$, where μ is the dielectric constant, B is the constant intensity of the magnetic field and σ_z is an operator in a 2-dimensional Hilbert space, which is represented by the Pauli matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in an ordered pair (ξ_+, ξ_-) of orthonormalized vectors. Consider the unit vector $\psi(t) = (\cos\theta)\xi_+ + (\mathrm{e}^{-\mathrm{i}\omega t}\sin\theta)\xi_-$, with $\omega = \mu B$ and $\theta \in \mathbb{R}$. The closed curve $\wp$, where $\wp(t)$ is the projection on $\widehat{\psi(t)}$ and $t \in [0, \frac{2\pi}{\omega}]$, is an invariant of H for every θ, but H is geometric for $\wp$ only if $\theta = \pm\frac{\pi}{4} \pmod{\pi}$. In these cases the Hamiltonian is off-diagonal with respect to the initial projections, the projections on the orthogonal subspaces spanned by $\psi_\pm = \frac{1}{\sqrt{2}}(\xi_+ \pm \xi_-)$, respectively. Then $\wp$ is a closed geodesic, the phase is geometric and is simply -1. If the initial state is chosen in an energy level of H, the invariant $\wp$ is constant; for general values of θ, a dynamic phase is added to the geometric phase.

The dynamical implementation of a geodesic arc in $\mathcal{P}_K(\mathfrak{H})$ with a constant Hamiltonian H requires us to choose an initial projection P such that H is P-off-diagonal. This means that the chosen initial states, the states in $\hat{E}$ with $E = \mathrm{Im}\,P$, cannot be in a definite energy level. In applications of non-Abelian geometric phases in quantum computation the eigenstates of the lowest energy level are still considered as initial states, but the geometric phases are just obtained in the adiabatic approximation [37]. Here we studied how to generate geometric phases by means of a Hamiltonian system without approximations. This can be interesting also in applications since, as stressed in [38], the adiabatic approach can have many disadvantages in physical implementations, due to the long evolution time necessary for an adiabatic process.

6. Holonomies arising from geodesic loops

In this section we shall study the holonomies of the connection $\mathcal{A}$ w.r.t. geodesic loops, i.e. loops consisting of geodesic arcs. Throughout this section we shall assume that K is a non-trivial closed subspace of $\mathfrak{H}$ and P is the orthogonal projection on K.

Lemma 3. *Let $W \in \mathfrak{p}$, $z = W|K$ and $A = -\mathrm{i}W$. Then $\mathrm{e}^W = \cos A + \mathrm{i}\sin A$ with*

$$\cos A = (\cos|z|)P + (\cos|z^\dagger|)P^\perp$$

and

$$\sin A = \mathrm{i}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(-z|z|^{2n}P + z^\dagger|z^\dagger|^{2n}P^\perp\right).$$

If z is invertible, then

$$(\sin A)|K = -\mathrm{i}z|z|^{-1}\sin|z| \quad \textit{and} \quad (\sin A|)K^{\perp} = \mathrm{i}z^{\dagger}|z^{\dagger}|^{-1}\sin|z^{\dagger}|.$$

Proof. By $A = -\mathrm{i}zP + \mathrm{i}z^{\dagger}P^{\perp}$ we get $A^2 = |A|^2 = |z|^2P + |z^{\dagger}|^2P^{\perp}$ so that $A^{2n} = |z|^{2n}P + |z^{\dagger}|^{2n}P^{\perp}$ and $A^{2n+1} = -\mathrm{i}z|z|^{2n}P + \mathrm{i}z^{\dagger}|z^{\dagger}|^{2n}P^{\perp}$. The formulae follow immediately. □

We recall that, if W is compact, then also z and $z^{\dagger}$ are compact. There exist orthonormal sequences $\{v_j\}_{j\in J} \subset K$ and $\{w_j\}_{j\in J} \in K^{\perp}$ such that

$$z(v) = \sum_{j\in J} s_j(v_j|v)w_j \quad \text{and} \quad |z|(v) = \sum_{j\in J} s_j(v_j|v)v_j \quad \text{for } v \in K$$

and

$$z^{\dagger}(w) = \sum_{j\in J} s_j(w_j|w)v_j \quad \text{and} \quad |z^{\dagger}|(w) = \sum_{j\in J} s_j(w_j|w)w_j \quad \text{for } w \in K^{\perp},$$

where by s_j we denote the singular values of z (for the spectral theory of compact operators, see e.g. [34]).

Lemma 4. *Let z be a compact operator. We have (in the above notation)*

$$(\sin A)|K = -\mathrm{i}\sum_{j\in J}\sin(s_j)(v_j|.)w_j$$

$$(\sin A)\,|K^{\perp} = \mathrm{i}\sum_{j\in J}\sin(s_j)(w_j|.)v_j.$$

We also have

$$(\cos A)|K = \cos|z| = \sum_{j\in J}\cos(s_j)(v_j|.)v_j + \Pi'$$

and

$$(\cos A)|K^{\perp} = \cos|z^{\dagger}| = \sum_{j\in J}\cos(s_j)(w_j|.)w_j + \Omega'$$

where Π and Ω denote the projections on the closed subspaces spanned by the families $\{v_j\}_{j\in j}$ and $\{w_j\}_{j\in J}$, respectively, and $\Pi' = \mathbf{1}_K - \Pi$, $\Omega' = \mathbf{1}_{K^{\perp}} - \Omega$.

Proof. The proof is just a standard computation. □

Which geodesic arcs are loops? For the next proposition it is worthwhile to recall that $R \in U(K)$ is called a *reflection of* K if $R = R^{\dagger}$. Equivalently, there exist Π and Π', orthogonal projections in K, such that $\Pi + \Pi' = \mathbf{1}_K$ and that $R = \Pi - \Pi'$.

Proposition 12. *Let K, P, $\mathfrak{p}$, W, A and z be as in Lemma 3. The following statements hold:*

1. *the geodesic $\wp(t) = \mathrm{e}^{Wt}P\mathrm{e}^{-Wt}$ is closed if and only if the spectrum of $|z|$ consists of finitely many eigenvalues $\{\lambda_1, \ldots, \lambda_r\}$ and the eigenvalues have rational ratios;*
2. *if a geodesic arc $\wp$ is a loop at P and $u_0 \in \mathrm{St}(K, \mathfrak{H})$ is over P, the relative holonomy $\mathrm{Hol}(\mathcal{A}, \wp, u_0)$ is a reflection $R = \Pi - \Pi'$ of K with $\mathrm{Rank}(\Pi') \le \dim(K^{\perp})$;*
3. *for every reflection $R = \Pi - \Pi'$ of K with $\mathrm{Rank}(\Pi') \le \dim(K^{\perp})$ and u_0 over P there exists a closed geodesic arc $\wp$ such that R is the relative holonomy $\mathrm{Hol}(\mathcal{A}, \wp, u_0)$.*

Proof. 1. As in Lemma 3 we put $\mathrm{e}^{tW} = \cos(tA) + \mathrm{i}\sin(tA)$. Therefore $\wp$ is closed if and only if a $\tau \in \mathbb{R}$ exists which satisfies

$$\sin(\tau A)v = 0 \quad \forall v \in K.$$

Suppose that z is invertible. By the identity

$$\sin(tA)v = -\mathrm{i}z|z|^{-1}\sin(t|z|) \quad \forall t \in \mathbb{R}, \forall v \in K$$

we are reduced to find τ such that

$$\sin(\tau|z|) = 0. \tag{6.1}$$

Let us denote by σ the spectrum of $|z|$ and by μ its projection-valued spectral measure. Then $\sin(|z|\tau) = 0$ if and only if $\mu(\cup_{k\in\mathbb{Z}}\{\lambda \in \sigma, \lambda\tau = k\pi\}) = \mu(\sigma) = \mathbf{1}_K$. As $\cup_{k\in\mathbb{Z}}\{\lambda \in \sigma, \lambda t = k\pi\}$ is at most countable for any t and $\mu\{\lambda\} = 0$ if λ is a point of the continuous spectrum, Eq. (6.1) is satisfied if and only if $|z|$ has at most countably many eigenvalues and a complete set of eigenvectors. Moreover, only the case of finitely many eigenvalues is allowed. Let $\{\lambda_l\}$ be an infinite set of distinct eigenvalues of $|z|$ and suppose that there exists τ such that the equations $\lambda_l\tau = k_l\pi$ admit integer solutions k_l for every l; then, without loss of generality, $\{k_l\}$ is a sequence of distinct integers such that $|k_l| \to +\infty$. As z is bounded, we have a contradiction. Let now $\{\lambda_1, \ldots, \lambda_r\}$ be the eigenvalues of $|z|$. It is well known that one can find τ satisfying $\lambda_l\tau = k_l\pi$ for every $l = 1, \ldots, r$ if and only if the eigenvalues have rational ratios.

The case of $\ker|z| \neq \{0\}$ can be reduced to the previous one since, by Lemma 3, the equation $\sin(tA)v = 0$ (for every t) is trivially satisfied for $v \in \ker|z|$.

2. Let now τ be a solution of Eq. (6.1)) and let u_K denote the canonical inclusion of K in $\mathfrak{H}$. Then $\mathrm{Hol}(\mathcal{A}, \wp, u_K)$ is simply given by $\cos(\tau|z|)$. Let $\{e_j\}_{j\in J}$ be a complete set of eigenvectors of $|z|$ such that $|z|e_j = \lambda_{e_j}e_j$ for every $j \in J$. Then $\cos(\tau|z|)e_j = \cos(\tau\lambda_{e_j})e_j = (-1)^{k_{e_j}}e_j$ if $\tau\lambda_{e_j} = k_{e_j}\pi$. Thus $\cos(\tau|z|)$ is a reflection of K. Let $u_0 \in \mathrm{St}(K, \mathfrak{H})$ be any isometric embedding over P. Then $\mathrm{Hol}(\mathcal{A}, \wp, u_0)$ is given by $u_0^\dagger\cos(\tau|z|)u_0$ and it is again a reflection.

To prove the last point, we have only to examine the case when $\dim(K^\perp) = r$ is finite. Then $\mathrm{Rank}(z) = \mathrm{Rank}(|z|) = d \leq r$. We can write $|z| = \sum_{j=1}^d s_j(e_j|v)e_j$ where $\{e_j\}$ is a suitable complete orthonormal system in K and the scalars s_j are strictly positive. Then

$$(\cos z)\, v = \sum_{j=1}^{d}\cos s_j(e_j|v)e_j + \sum_{j>d}(e_j|v)e_j \quad \forall v \in K.$$

To get $\cos z = \Pi - \Pi'$ it is necessary that $\mathrm{Rank}(\Pi') \leq d \leq r$.

3. Assume $\mathrm{Rank}(\Pi') \leq \dim(K^\perp)$ and $R = \Pi - \Pi'$. We prove that there exists $z \in L(K, K^\perp)$ such that $\cos|z| = R$. Let $\{e_j\}_{j\in J}$ be an orthonormal system in K such that the closed subspace spanned by $\{e_j\}_{j\in J}$ agrees with $\mathrm{Im}\,\Pi'$ and let $\{w_j\}_{j\in J}$ be an orthonormal system in $K^\perp$. We define, for $v \in K$

$$z(v) := \sum_{j\in J}\pi(e_j|v)w_j.$$

Since $z^\dagger z = \pi^2\Pi'$, we get $|z| = \pi\Pi'$ and $\cos|z| = R$, as required. $\square$

Let us consider the loops based at P which are products of closed geodesic arcs. Their holonomies form the group generated by the reflections of K with the property in point 2 of Proposition 12. This group does not depend on the $u_0 \in \mathrm{St}(K, \mathfrak{H})$ which has been chosen. In particular, if $\dim(K) \leq \dim(K^\perp)$, we obtain the group generated by all reflections of K. If $\dim(K)$ is finite one can easily see that this is a proper subgroup of $U(K)$ since its elements have real determinant.

More generally, we call *(piecewise) geodesic loop* any loop in $\mathcal{P}_K(\mathfrak{H})$ which is the product of geodesic arcs. For $u_0 \in \mathrm{St}(K, \mathfrak{H})$ over P, we are interested in the group of the holonomies at u_0 which are generated by geodesic loops based on P. We call this group the *geodesic holonomy group at* u_0 and denote it by $\mathrm{Hol}_{\mathrm{geod}}(\mathcal{A}, u_0)$.

Proposition 13. *Let $W_1 = \mathrm{i}A_1$ and $W_2 = \mathrm{i}A_2$ belong to $\mathfrak{p}$ with $W_1|K = z_1$ and $W_2|K = z_2$ and consider the geodesic arcs $\wp_1(t) = \mathrm{e}^{tW_1}P\mathrm{e}^{-tW_1}$ for $0 \leq t \leq t_1$ and $\wp_2(t) = \mathrm{e}^{-tW_2}P\mathrm{e}^{tW_2}$ for $0 \leq t \leq t_2$. The following conditions are equivalent:*

1. $\wp_1(t_1) = \wp_2(t_2)$;
2. $(\sin(t_2A_2)\cos(t_1A_1) + \cos(t_2A_2)\sin(t_1A_1))\,|K = 0.$
3. $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(t_2^{2n+1}z_2|z_2|^{2n}\cos|t_1z_1| + t_1^{2n+1}\cos(t_2|z_2^\dagger|)z_1|z_1|^{2n}\right) = 0.$
4. *In the particular case that $|z_1|$ and $|z_2|$ are invertible the above condition 3 becomes*

$$z_2|z_2|^{-1}\sin(t_2|z_2|)\cos(t_1|z_1|) + \cos(t_2|z_2^\dagger|)z_1|z_1|^{-1}\sin|t_1z_1| = 0.$$

If one of these conditions is satisfied, then $\wp_2^{-1} \circ \wp_1$ *is a geodesic loop and*

$$\mathrm{Hol}(\wp_2^{-1} \circ \wp_1, \mathcal{A}, u_K) = (\cos(t_2 A_2)\cos(t_1 A_1) - \sin(t_2 A_2)\sin(t_1 A_1))\,|K$$

where u_K *denotes the canonical inclusion of* K *in* $\mathfrak{H}$.

Proof. $1 \iff 2$. Condition 1 is satisfied if and only if $\mathrm{e}^{t_2 W_2}\mathrm{e}^{t_1 W_1} P \mathrm{e}^{-t_1 W_1}\mathrm{e}^{-t_2 W_2} = P$. The product $\mathrm{e}^{t_2 W_2}\mathrm{e}^{t_1 W_1}$ is given by

$$(\cos(t_2 A_2)\cos(t_1 A_1) - \sin(t_2 A_2)\sin(t_1 A_1)) + \mathrm{i}\,(\sin(t_2 A_2)\cos(t_1 A_1) + \cos(t_2 A_2)\sin(t_1 A_1))$$

where the first term is P-diagonal and the second one is P-off-diagonal. Therefore $e^{t_2 W_2} e^{t_1 W_1}$ preserves K if and only if $(\sin(t_2 A_2)\cos(t_1 A_1) + \cos(t_2 A_2)\sin(t_1 A_1))\,|K = 0$.

$2 \iff 3$. Using Lemma 3, condition 2 can be written

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(t_2^{2n+1} z_2 |z_2|^{2n} \cos(t_1|z_1|) + t_1^{2n+1}\cos(t_2|z_2^{\dagger}|) z_1 |z_1|^{2n}\right) = 0.$$

4. This statement follows immediately by Lemma 3.

For the last statement we observe that the final point of the horizontal lift of $\wp_2^{-1} \circ \wp_1$ is given by

$$\mathrm{e}^{t_2 W_2}\mathrm{e}^{t_1 W_1} u_K = \mathrm{e}^{t_2 W_2}\mathrm{e}^{t_1 W_1}|K = u_K \mathrm{e}^{t_2 W_2}\mathrm{e}^{t_1 W_1}|K$$

since u_K is the canonical inclusion. □

Proposition 14. *Let* K *be a subspace with* $1 \le \dim(K) \le \dim(K^{\perp})$. *Then every* $S \in U(K)$ *is the holonomy of a loop at* P *composed by two geodesic arcs.*

Proof. Let $S \in U(K)$ and choose $z_1 \in L(K, K^{\perp})$ with $|z_1| = \mathbf{1}_K$ and $z_2 = z_1 S^{\dagger}$. Thus $|z_2| = S|z_1|S^{\dagger} = \mathbf{1}_K$ and $|z_2^{\dagger}| = |z_1^{\dagger}| = \Omega$ where Ω denotes the projection on the range of z_1. Let W_1 and W_2 be in $\mathfrak{p}$ such that $W_1|K = z_1$ and $W_2|K = z_2$ and consider the curves $\wp_1$ and $\wp_2$ as in Proposition 13 with $t_1 = \frac{\pi}{2}$ and $t_2 = \frac{3}{2}\pi$. Since $\cos(t_1|z_1|) = 0$ and $\cos(t_2|z_2^{\dagger}|) = \Omega' := \mathbf{1}_{K^{\perp}} - \Omega$, point 4 of Proposition 13 assures that the curve $\wp_2^{-1}\wp_1$ is a loop. Since $\cos(t_1 A_1)|K = \cos(t_1|z_1|) = 0$, the holonomy of this loop is $-\sin(t_2 A_2)\sin(t_1 A_1)|K$ and equals

$$-(\sin t_1) z_2^{\dagger}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!} t_2^{2n+1}\Omega^{2n} z_1\right) = -(\sin t_1)(\sin t_2) z_2^{\dagger} z_1 = z_2^{\dagger} z_1 = S$$

as required. □

Lemma 5. *Assume* $1 \le \dim(K^{\perp}) < \dim(K)$ *and let* $T \in \mathfrak{u}(K)$ *be a finite rank operator. Then* $S := e^T$ *belongs to the geodesic holonomy group* $\mathrm{Hol}_{\mathrm{geod}}(\mathcal{A}, u_K)$.

Proof. We denote by r the finite dimension of $K^{\perp}$. Using the spectral decomposition of T we can factorize $S = \prod_{i=1}^{n} S_i$ where $S_i = e^{T_i}$ with $T_i \in \mathfrak{u}(K)$ with $\mathrm{rank}(T_i) \le r$. Hence it is not restrictive to assume that $\mathrm{rank}(T) \le r$. We choose an orthonormal basis $\{w_j\}_{j=1,\dots,r}$ in $K^{\perp}$ and an orthonormal family $\{v_j\}_{j=1,\dots,r}$ in K and define $z_1 \in L(K, K^{\perp})$ by $z_1 = \sum_{j=1}^{r}(v_j|.)w_j$. Then $|z_1| = \Pi$, where Π is the orthogonal projection on $\mathrm{Span}\{v_j\}$ and $|z_1^{\dagger}| = \mathbf{1}_{K^{\perp}}$. One can arrange the family $\{v_j\}_{j=1,\dots,r}$ so that $\Pi S = S\Pi$ and $S\Pi' = \Pi'$. Let $z_2 = z_1 S^{\dagger}$, hence $|z_2| = \Pi$ and $|z_2^{\dagger}| = \mathbf{1}_{K^{\perp}}$. Consider now $\wp_1$ and $\wp_2$ as in Proposition 13, with $W_1|K = z_1$ and $W_2|K = z_2$, $t_1 = \pi/2$ and $t_2 = \frac{3}{2}\pi$. Then condition 2 in Proposition 13 is verified. Actually, $\cos(t_2 A_2)w = \cos|t_2 z_2^{\dagger}|w = \cos(\frac{3}{2}\pi \mathbf{1}_{K^{\perp}})w = 0$ for every $w \in K^{\perp}$ and $\sin(t_2 A_2)\cos(t_1 A_1)v = \sin(t_2 A_2)\Pi' v = \mathrm{i}\sum_{j=1}^{r}(Sv_j|\Pi' v)w_j = 0$ for every $v \in K$.

Let us calculate the holonomy relative to the loop $\wp_2^{-1}\circ\wp_1$, by applying the formula in Proposition 13. By Lemma 4, we have $\cos(t_2 A_2)\cos(t_1 A_1)v = \Pi' v$, for $v \in K$. Moreover, from

$$\sin(t_1 A_1)v = -\mathrm{i}\sum_{j=1}^{r}\sin(t_1)(v_j|v)w_j = -\mathrm{i}\sum_{j=1}^{r}(v_j|v)w_j$$

and from

$$\sin(t_2 A_2)w = \mathrm{i}\sum_{j=1}^{r} \sin(t_2)(w_j|w)Sv_j$$

we get $\sin(t_2A_2)\sin(t_1A_1)v = -S\Pi v$ for every $v \in K$. We conclude that

$$\mathrm{Hol}(\wp_2^{-1} \circ \wp_1, \mathcal{A}, u_K)v = \Pi' v + S\Pi v = Sv$$

as required. □

We obtain immediately the following proposition.

Proposition 15. *Assume* $\dim(\mathfrak{H})$ *finite or* $\dim(K) \leq \dim(K^\perp)$. *Then*

$$\mathrm{Hol}_{\mathrm{geod}}(\mathcal{A}, u_0) = \mathrm{Hol}(\mathcal{A}, u_0) = U(K)$$

for every $u_0 \in \mathrm{St}(K, \mathfrak{H})$.

We want to investigate the geodesic holonomy group of the connection $\mathcal{A}$ in the remaining case where $\dim(K)$ is infinite and $\dim(K^\perp)$ is finite. We recall that $U(\mathfrak{H})$ is contractible whenever $\dim(\mathfrak{H})$ is infinite [16]. This implies that $\mathcal{P}_K(\mathfrak{H}) \simeq U(\mathfrak{H})/U_K(\mathfrak{H})$ is simply connected, so that the holonomy group is connected and agrees with the restricted holonomy group. In the theory of Banach Lie principal bundles however, also the restricted holonomy group can fail to be a Lie subgroup in our strong sense. In the next proposition we prove that the holonomy group, in the case under consideration, is a subgroup of $U_\infty(\mathfrak{H})$, a Banach Lie group which is a closed subgroup of $U(\mathfrak{H})$ but not a Lie subgroup. Let us denote by $L_\infty(\mathfrak{H})$ the closed ideal of the C^*-algebra $L(\mathfrak{H})$ consisting of the compact operators. By $U_\infty(\mathfrak{H})$ we denote the unitary Fredholm group, i.e. the group of the unitary operators U of the form $U = \mathbf{1}_{\mathfrak{H}} + X$ with $X \in L_\infty(\mathfrak{H})$. It is well known that $U_\infty(\mathfrak{H})$ is a Banach Lie group whose Banach Lie algebra is $u_\infty(\mathfrak{H}) := \{X \in L_\infty(\mathfrak{H}) \ : \ X^\dagger = -X\}$. We stress that $u_\infty(\mathfrak{H})$ is not a splitting subalgebra of $u(\mathfrak{H})$ so that $U_\infty(\mathfrak{H})$ is not a Lie subgroup of $U(\mathfrak{H})$. The exponential map $\exp : \mathfrak{u}_\infty(\mathfrak{H}) \to U_\infty(\mathfrak{H})$ is onto, hence $U_\infty(\mathfrak{H})$ is connected. For details, see [13].

Proposition 16. *Assume that K is an infinite dimensional non-trivial subspace of $\mathfrak{H}$ and that $K^\perp$ is finite dimensional. Then*

$$\mathrm{Hol}(\mathcal{A}, u_0) \subset U_\infty(K)$$

for $u_0 \in \mathrm{St}(K, \mathfrak{H})$ *over* P.

Proof. Let $\wp$ be a loop with base point P. Using the small lassos technique (see Appendix 7 in [15]) one can represent $\wp$ as a product of loops of the form $\mu^{-1} \circ \ell \circ \mu$ where ℓ is a loop which lies completely in a chart and μ is a path which joins the base point P with the base point, say P_ℓ, of the loop ℓ. By $\mathsf{Pt}^{\mathcal{A}}(\mu^{-1} \circ \ell \circ \mu)(u_0) = u_0 \mathrm{Hol}(\mu^{-1} \circ \ell \circ \mu)$ we get

$$\mathrm{Hol}(\mathcal{A}, \mu^{-1} \circ \ell \circ \mu, u_0) = \mathrm{Hol}(\mathcal{A}, \ell, \mathsf{Pt}^{\mathcal{A}}(\mu)(u_0)).$$

We have only to prove that holonomies relative to loops which lie completely in a chart belong to $U_\infty(K)$. Let now $\mathcal{U}_P$ and β_P be as in Lemma 2 and let $\wp : [0, 1] \to \mathcal{P}_K(\mathfrak{H})$ be a loop at P contained in $\mathcal{U}_P$. Set $z(t) := \beta_P(\wp(t))$, so that $z(0) = z(1) = 0$. For every $u_0 \in \mathrm{St}(K, \mathfrak{H})$ over P, the curve $\hat{v}(t) := u_0 + z(t)u_0$ is a lift of $\wp$ in $\mathrm{Emb}(K, \mathfrak{H})$, so that $v(t) := \hat{v}(t)|\hat{v}(t)|^{-1}$ is a lift of $\wp$ in $\mathrm{St}(K, \mathfrak{H})$ starting and ending in u_0. A simple computation shows that $\mathrm{rank}(\dot{v}(t)) \leq r$ for every t, $r = \dim(K^\perp)$. Therefore also the rank of $\mathcal{A}(\dot{v}(t)) = v^\dagger(t)\dot{v}(t)$ is not greater than r.

The horizontal lift of $\wp$ is $u(t) = v(t)S(t)$ where $S(t)$ is a C^1 curve in $U(K)$ which satisfies the equation

$$\dot{S}(t)S^\dagger(t) = -\mathcal{A}(\dot{v}(t)), \qquad S(0) = \mathbf{1}_K.$$

Since $\mathcal{A}(\dot{v}(t)) \in \mathfrak{u}_\infty(\mathfrak{H})$, the solution $S(t)$ of the above equation belongs to $U_\infty(K)$ for every t. Finally, we have only to recall that $\mathrm{Hol}(\mathcal{A}, \wp, u_0) = S(1)$. □

From Lemma 5 and Proposition 16 we obtain that, if $\dim(K)$ is infinite and $\dim(K)^{\perp}$ is finite, the following inclusions hold:

$$\mathrm{Hol}_{\mathrm{geod}}(\mathcal{A}, u_K) \subset \mathrm{Hol}(\mathcal{A}, u_K) \subset U_{\infty}(K),$$

with $\overline{\mathrm{Hol}_{\mathrm{geod}}(\mathcal{A}, u_K)} = U_{\infty}(K)$.

Thus we can conclude that the geodesic holonomy group agrees with $U(K)$ if K is finite dimensional or if $\dim(K) \leq \dim(K^{\perp})$ and we obtain that the holonomy group is $U(K)$, extending a well known result. Actually, in the case where $\mathfrak{H}$ is finite dimensional, one can prove that the holonomy group agrees with $U(K)$ by using the Ambrose Singer theorem, a theorem which is difficult to extend to infinite dimensions (compare, e.g., [20] and [33]). However, in the case where K is finite dimensional and $\mathfrak{H}$ is infinite dimensional one can prove that the holonomy group is $U(K)$ by using an extended version of the Ambrose Singer theorem (see [12]).

In the remaining critical case, where $K^{\perp}$ is finite dimensional and K is infinite dimensional, we proved that the holonomy group is contained in the unitary Fredholm group $U_{\infty}(K)$. However, using the holonomies relative to geodesic loops, one can approximate a generic operator in $U_{\infty}(K)$.

Acknowledgment

The authors would like to thank the referee for many useful suggestions and remarks and for the care with which the manuscript has been reviewed.

Appendix

Distances for topology on projections and subspaces. Consider the projective space $\mathbf{P}(\mathfrak{H}) = \mathrm{Gr}_{\mathbb{C}}(\mathfrak{H})$. As is well known, the projective space has a canonical structure of Kähler manifold. The geodesical properties of $\mathbf{P}(\mathfrak{H})$ are quite simple, see e.g. [7]. We recall that $\mathbf{P}(\mathfrak{H})$ consists of the 1-dimensional subspaces $\hat{e}$ (called rays) spanned by all $e \in \mathfrak{H}$ with $\|e\| = 1$. The geodesic joining two different rays $\hat{e}$ and $\hat{f}$ belongs to the projective space of the 2-dimensional subspace spanned by e and f. Since this projective space is isometrically isomorphic to the unit sphere S^2 equipped with the riemannian metric induced by the euclidean metric on $I\mathbb{R}^3$, geodesics describe great circles and, therefore, the Kähler distance $\mathsf{d}(\hat{e}, \hat{f})$ is defined as the length of the great circle arc joining $\hat{e}$ and $\hat{f}$ [7]. An easy computation shows that

$$|(e|f)| = \cos(\frac{1}{2}\mathsf{d}(\hat{e}, \hat{f})). \tag{A.1}$$

Each non-zero subspace E of $\mathfrak{H}$ is canonically identified with the closed subset $\hat{E}$ of $\mathbf{P}(\mathfrak{H})$ of all rays $\hat{e}$ with $e \in E$. So we can consider the Hausdorff distance between $\hat{E}$ and $\hat{F}$ for non-zero subspaces E and F. We define

$$\mathsf{dist}(\hat{e}, \hat{F}) := \inf\{\mathsf{d}(\hat{e}, \hat{f}) \mid \hat{f} \in \hat{F}\} \quad \text{and} \quad \mathsf{D}_0(\hat{E}, \hat{F}) := \sup_{\hat{e} \in \hat{E}} \mathsf{dist}(\hat{e}, \hat{F})$$

and the distance D on $\mathrm{Gr}(\mathfrak{H})$ by

$$\mathsf{D}(E, F) := \max\{\mathsf{D}_0(\hat{E}, \hat{F}), \mathsf{D}_0(\hat{F}, \hat{E})\}.$$

There are many geometric invariants related to incidence properties of subspaces. A careful discussion of these invariants for subspaces of Banach spaces can be found in [25]. Here we restrict our attention to Hilbert spaces. The problem arises to characterize the incidence properties of subspaces in terms of the distance D and of the norm distance between the associated orthogonal projections. Other distances on projections are discussed in [4,27].

Let $\mathfrak{H}$ be a complex Hilbert space. For $x \in \mathfrak{H}$ and for a closed non-empty subset Γ of $\mathfrak{H}$, we denote

$$\mathrm{dist}(x, \Gamma) := \inf\{\|v - x\| \quad v \in \Gamma\}.$$

For a subspace E one has $\mathrm{dist}(x, E) = \|P^{\perp}x\|$, with P the projection operator on E. We set

$$S(E) := \{e \in E \mid \|e\| = 1\} \quad \text{and} \quad B(E) := \{b \in E \mid \|b\| \leq 1\}.$$

Lemma 6. *Let E be a non-zero subspace of $\mathfrak{H}$ and let P denote the orthogonal projection on E. For every $x \in \mathfrak{H}$,*

$$\|Px\| = \sup_{e \in S(E)} |(e|x)| = \sup_{b \in B(E)} |(b, x)|.$$

Proof. Let $e \in S(E)$ and let p be the 1-dimensional projection operator on the 1-dimensional subspace generated by e. By $P \geq p$ one gets $\|Px\|^2 \geq \|px\|^2 = |(e|x)|^2$, so that

$$\sup_{e \in S(E)} |(e|x)| \leq \|Px\| \leq \sup_{b \in B(E)} |(b|x)|.$$

For every $b \in B(E)$ with $0 < \|b\| = \beta$, consider $e := \beta^{-1} b$. Then $e \in S(E)$ with

$$|(e|x)| = \beta^{-1} |(b|x)| \geq |(b|x)|.$$

This implies our statement. $\square$

By Lemma 6 we obtain that

$$\operatorname{dist}(x, E) = \|P^{\perp} x\| = \sup_{e \in S(E^{\perp})} |(e|x)| = \sup_{b \in B(E^{\perp})} |(b|x)|.$$

Let E and F be non-zero subspaces of $\mathfrak{H}$ and define

$$\Theta_0(E, F) := \sup_{e \in S(E)} \operatorname{dist}(e, F).$$

The quantity

$$\Theta(E, F) := \max\{\Theta_0(E, F), \Theta_0(F, E)\}$$

is called the *opening (or aperture) between E and F* [34].

Lemma 7. *Let E and F be non-zero subspaces of $\mathfrak{H}$. Then*

$$\Theta_0(E, F) = \sin\left(\frac{1}{2} \mathsf{D}_0(\hat{E}, \hat{F})\right).$$

Proof. For $e \in S(E)$ we have

$$\operatorname{dist}^2(e, F) = 1 - \|Qe\|^2 = 1 - \sup_{u \in S(F)} |(u|e)|^2 = \inf_{u \in S(F)} (1 - |(u|e)|^2)$$

so that, by formula (A.1), can be found

$$\operatorname{dist}(e, F) = \inf_{u \in S(F)} \sin \frac{1}{2} \mathsf{d}(\hat{e}, \hat{u}) = \sin \frac{1}{2} \mathsf{dist}(\hat{e}, \hat{F}).$$

Therefore

$$\Theta_0(E, F) = \sup_{e \in S(E)} \operatorname{dist}(e, F) = \sup_{e \in S(E)} \sin \frac{1}{2} \mathsf{dist}(\hat{e}, \hat{F}) = \sin \frac{1}{2} \mathsf{D}_0(\hat{E}, \hat{F}). \quad \square$$

Lemma 8. *Let P and Q be orthogonal projections in $\mathfrak{H}$. Then*

$$\|P - Q\|^2 = \max\{\|(P - Q)^2 P\|, \|(P - Q)^2 P^{\perp}\|.$$

Proof. We denote $\max\{\|(P - Q)^2 P\|, \|(P - Q)^2 P^{\perp}\|\}$ by M. The norm properties give

$$\|(P - Q)^2 R\| \leq \|(P - Q)^2\| = \|P - Q\|^2$$

(where R denotes P or $P^{\perp}$) so that $M \leq \|P - Q\|^2$. Conversely, for $x \in S(\mathcal{H})$ we have

$$\begin{aligned}(x|(P - Q)^2 x) &= (Px|(P - Q)^2 Px) + (P^{\perp} x|(P - Q)^2 P^{\perp} x) \\ &\leq \|(P - Q)^2 P\| \|Px\|^2 + \|(P - Q)^2 P^{\perp}\| \|P^{\perp} x\|^2 \leq M\end{aligned}$$

so that

$$\|P-Q\|^2 = \|(P-Q)^2\| = \sup_{x\in S(\mathcal{H})} (x|(P-Q)^2x) \leq M. \quad \square$$

We look now for a relation between the opening $\Theta(E, F)$ and the spectrum of PQP. The spectrum of an operator A will be denoted by σ_A.

Lemma 9. *Let E and F be non-zero subspaces of $\mathfrak{H}$ and let P and Q denote the orthogonal projections on E and F, respectively. The following statements hold.*

1. $(P-Q)^2P = P(P-Q)^2 = PQ^{\perp}P$.
2. $\|PQ^{\perp}P\| = \|PQ^{\perp}|E\| = 1 - \inf \sigma_{PQ|E}$.
3. $\|(P-Q)^2P\| = \|PQ^{\perp}P\| = \Theta_0^2(E, F)$.

Proof. 1. It is trivial.

2. We have $\|(P-Q)^2P\| = \|PQ^{\perp}P\| \geq \|PQ^{\perp}|E\| = \sup \sigma_{PQ^{\perp}|E} = 1 - \inf \sigma_{PQ|E}$. To prove the converse relation, recall that

$$\|PQ^{\perp}P\| = \sup_{x\in S(\mathcal{H})} (x|PQ^{\perp}Px).$$

Assume that $0 < \|Px\| = \beta < 1$ and define $e := \beta^{-1}Px$. Then $e \in S(E)$ and

$$(e|PQ^{\perp}e) = (e|Q^{\perp}e) = \beta^{-2}(x|PQ^{\perp}Px) \geq (x|PQ^{\perp}Px).$$

This implies $\|PQ^{\perp}|E\| \geq \|PQ^{\perp}P\|$, as required.

3. It follows by

$$\Theta_0^2(E, F) = \sup_{e\in S(E)} \|Q^{\perp}e\|^2 = \sup_{e\in S(E)} (e|PQPe) = \|PQ^{\perp}P\|. \quad \square$$

We obtain immediately the following proposition.

Proposition 17. *Let E and F be non-zero subspaces of $\mathfrak{H}$ and let P and Q denote the orthogonal projections on E and F, respectively. Then*

$$\|P-Q\| = \Theta(E, F) = \sin\left(\frac{1}{2}\mathsf{D}(E, F)\right).$$

By the above proposition the natural map $\mathcal{P} : \mathrm{Gr}(\mathfrak{H}) \to \mathcal{P}(\mathfrak{H})$, $E \mapsto \mathcal{P}_E$ is a homeomorphism. In particular, it restricts to a homeomorphism of $\mathrm{Gr}_K(\mathfrak{H})$ with $\mathcal{P}_K(\mathfrak{H})$. Other proofs of the first equality can be found in the literature, see e.g. [1].

Proposition 18. *Let E and F be non-zero subspaces of $\mathfrak{H}$ and let P and Q denote the orthogonal projections on E and F, respectively. Denote by $Q|E$ and $P|F$ the restrictions of Q to E and P to F, respectively. The following conditions are equivalent:*

1. $\mathsf{D}(E, F) < \pi$;
2. $\|P-Q\| < 1$;
3. $\inf \sigma_{PQ|E} > 0$ *and* $\inf \sigma_{P^{\perp}Q^{\perp}|E^{\perp}} > 0$;
4. $\inf_{e\in S(E)} \|Qe\| > 0$ *and* $\inf_{u\in S(F)} \|Pu\| > 0$;
5. $Q|E : E \to F$ *is a bijection;*
6. $P|F : F \to E$ *is a bijection;*
7. $PQ|E \in GL(E)$ *and* $QP|F \in GL(F)$;
8. *there exists a (unique)* $z \in L(E, E^{\perp})$ *such that* $F = \mathrm{graph}(z)$.

If $\dim(E) = \dim(F) < \infty$ *these conditions are equivalent to* $PQ|E \in GL(E)$.

Proof. Conditions 1 and 2 are equivalent by Proposition 17. Moreover 2 and 3 are equivalent by Proposition 17, by the above lemmas and by the definition of $\Theta(E, F)$.

2 $\iff$ 4. Actually, the inequality $\inf_{e\in S(E)} \|Qe\| > 0$ is equivalent to $\Theta_0(E, F) < 1$. Analogously for the other inequality.

Conditions 4, 5 and 6 are equivalent. The inequality $\inf_{e\in S(E)} \|Qe\| > 0$ implies that the bounded operator $Q|E$ is invertible, with bounded and closed inverse. Therefore its range is closed. The second inequality implies the analogous statement for P/F. By $(Q|E)^\dagger = P|F$ we obtain that $Q|E$ is surjective so that the equivalence of 4, 5 and 6 follows.

6 $\Rightarrow$ 7. By $PQ|E = (Q|E)^\dagger Q|E$ and by the equivalence of 5 and 6 we get $PQ|E \in GL(E)$. Analogously for $QP|F$.

7 $\Rightarrow$ 4. Since $PQ|E = (Q|E)^\dagger Q|E$, its numerical range is strictly positive, i.e. there exists $\mu > 0$ such that $(e|PQe) > \mu^2$ for every $e \in S(E)$ or, equivalently,

$$\inf_{e\in S(E)} \|Qe\| > \mu.$$

Analogously, $QP|F \in GL(F)$ implies $\inf_{u\in S(F)} \|Pu\| > 0$.

6 $\Rightarrow$ 8. Set $z := P^\perp(P|F)^{-1}$. Then $\mathrm{graph}(z) = F$. Actually, we can write each $f \in F$ as $f = x + P^\perp(P|F)^{-1}x$ with $x = Pf$. Thus, $F \subset \mathrm{graph}(z)$. Conversely, every $x \in E$ can be written as $x = Pf$ with $f \in F$, so that $x + zx = f$. Thus, $\mathrm{graph}(z) \subset F$.

8 $\Rightarrow$ 6. By $F = \mathrm{graph}(z)$ and $P(x + zx) = x$ for every $x \in E$, we get that $P : F \to E$ is a bijection.

If $\dim(E) = \dim(F) < \infty$, then one gets condition 6 simply by requiring that $PQ|E$ is injective. Actually, this condition implies that $Q|E$ is into and hence also onto F. □

Corollary 2. *The map $\beta_E : \mathcal{U}_E \to L(E, E^\perp)$, given by $\beta_E(F) = P^\perp(P|F)^{-1}$ is a homeomorphism and its inverse is the map $\alpha_E : L(E, E^\perp) \to \mathcal{U}_E$, $\alpha_E(z) = \mathrm{graph}(z)$.*

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